A Rapidly Convergent Expansion Method for Asian and Basket Options

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Outline

1. Asian and Related Option Payoffs
2. Existing Valuation Techniques
3. Characteristic Function Expansions
4. Numerical Convergence Properties
5. Theoretical Convergence Properties
6. Conclusions

1.1. Standard Asian Options

- Ubiquitious payoff structure
  - Set of sampling times: $\{t_i : i = 1, 2, ..., n\}$ (intraday, daily, weekly, monthly, yearly...) and a payoff date $T \geq t_n$
  - Fixings from asset price process: S_t: $\{S_{t_i}\}$
  - Set of known weights : $\{w_i\} : \sum_{i=1}^{n} w_i = 1$
  - Define arithmetic average: $A_T \equiv \sum_{i=1}^{n} w_i S_{t_i} = \bar{w} \cdot \bar{S}$
  - (Arithmetic) Asian (end) call payoff: $AC_T \equiv [A_T - K]^+$
- Uses
  - Hedge exposures distributed through time
  - Avoid impact of abnormal prices at expiry
  - Allow smooth unwind of hedge for cash-settled options
  - Asian put: $AP_T \equiv [K - A_T]^+$ (put-call parity via average of forwards)

1.1. Standard Asian Options (2)

- Valuation issues
  - Underlying driver:

    Brownian motion: $z_t \sim N(0,t)$

  - Asset price process:

    $$\frac{dS_t}{S_t} = (r_t - y_t)dt + \sigma_t dz_t = \mu_t dt + \sigma_t dz_t$$

    with r, y, σ functions of t only

  - "Black-Scholes+" framework (lognormal prices under risk-neutral measure)
  - Well-known results:
    - Sum of lognormals isn't lognormal
    - Only in very special cases (continuous sampling, constant parameters) is anything like closed-form solution possible

1.2. Related Payoffs

- Average strike (Asian start) options: $ASC_T = [S_T - kA_T]^+$
  - Smooth initial delta hedge for physically settled options
  - Change of (numeraire) measure: mappable into a standard Asian option

    $E[S_T - kA_T]^+ = E[S_T]\, E'[1 - kA'_T]^+$

    with E' an expectation and A'_T an average over a process running "backwards" from T to each t_i

- Asian start/Asian end options: $ASAEC_T = [A'_T - kA_T]^+$

  with $A'_T \equiv \sum_{i'=1}^{n'} {w'}_{i'} S_{{t'}_{i'}}$, $\{S_{{t'}_{i'}}\}$, and $\sum_{i'=1}^{n'} {w'}_{i'} = 1$
  - Avoid impact of abnormal prices at initiation and expiry
  - Smoothes management of both initial delta hedge and final unwind
  - Valuation: integration over joint distribution of two correlated averages
  - Also "fixed notional" payoff: $\left[\frac{A'_T}{A_T} - k\right]^+$

1.2. Related Payoffs (2)

- Basket options: $BC_T = [B_T - K]^+$

  with $B_T \equiv \sum_{j=1}^{m} w_j S_{j,T}$, $\sum_{j=1}^{m} w_j = 1$, and $S_{j,T}$ generated by:

  $$\frac{dS_{j,t}}{S_{j,t}} = (r_{j,t} - y_{j,t})dt + \sigma_{j,t} dz_{j,t} = \mu_{j,t} dt + \sigma_{j,t} dz_{j,t} : \ dz_{j,t} dz_{j',t} = \rho_{j,j',t}$$

  - Diversification effect: $\left[\sum_{j=1}^{m} w_j S_{j,T} - K\right]^+ \leq \sum_{j=1}^{m} w_j [S_{j,T} - K]^+$
  - Valuation: analogous to Asian option, but correlation structure not decomposable into independent increments

- Asian basket options: $ABC_T = [AB_T - K]^+$

  with $AB_T \equiv \sum_{i=1}^{n} \sum_{j=1}^{m} w_{i,j} S_{j,t_i}$, $\sum_{i=1}^{n} \sum_{j=1}^{m} w_{i,j} = 1$

  - Combination of average and basket features
  - Also available in AE, ASAE forms

1.2. Related Payoffs (3)

- Options on cash dividend-paying stocks
  - Consider price process modified to pay discrete cash dividends:

    $$\frac{dS_t}{S_t} = \left(r_t - y_t - \sum_{i=1}^{n} \frac{D_i}{S_t} \delta(t - t_i) \right) dt + \sigma_t dz_t$$

  - Consider also the (original) process with cash dividends reinvested:

    $$\frac{dR_t}{R_t} = (r_t - y_t)dt + \sigma_t dz_t : \; R_0 = S_0$$

  - It is easy to show that:

    $$S_{T>t_n} = R_T - \sum_{i=1}^{n} D_i \frac{R_T}{R_{t_i}}, \text{ hence } [S_T - K]^+ = \left[R_T - \sum_{i=1}^{n} D_i \frac{R_T}{R_{t_i}} - K \right]^+$$

  - Change of measure: mappable into Asian put on "reciprocal" process:

    $$E[S_T - K]^+ = E[R_T] \, E' \left[1 - \sum_{i=1}^{n} D_i R'_{t_i} - K R'_T \right]^+, \text{ with:}$$

    $$\frac{dR'_t}{R'_t} = (y_t - r_t)dt + \sigma_t dz'_t : \; R'_0 = \frac{1}{S_0}$$

1.2. Related Payoffs (4)

- Geometric average options: $GC_T \equiv [G_T - K]^+$ with $G_T \equiv \prod_{i=1}^{n} S_{t_i}^{w_i}$
  - Geometric averages (products) of lognormal variables are lognormal
  - Of interest primarily because most payoffs may be valued in closed form

- Harmonic average options: $HC_T \equiv [H_T - K]^+$ with $H_T \equiv \left(\sum_{i=1}^{n} \frac{w_i}{S_{t_i}} \right)^{-1}$
  - Of interest because of analogy to "fixed notional" payoffs

- Other average options: $A_pC_T \equiv \left[(A_p)_T - K\right]^+$ with $(A_p)_T \equiv \left(\sum_{i=1}^{n} w_i S_{t_i}^p \right)^{1/p}$
  - Contains all averages as subcases
    - $p = 1$: arithmetic; $p = 0$: geometric; $p = -1$: harmonic
    - $p = \infty$: maximum; $p = -\infty$: minimum

2.0. Existing Valuation Techniques

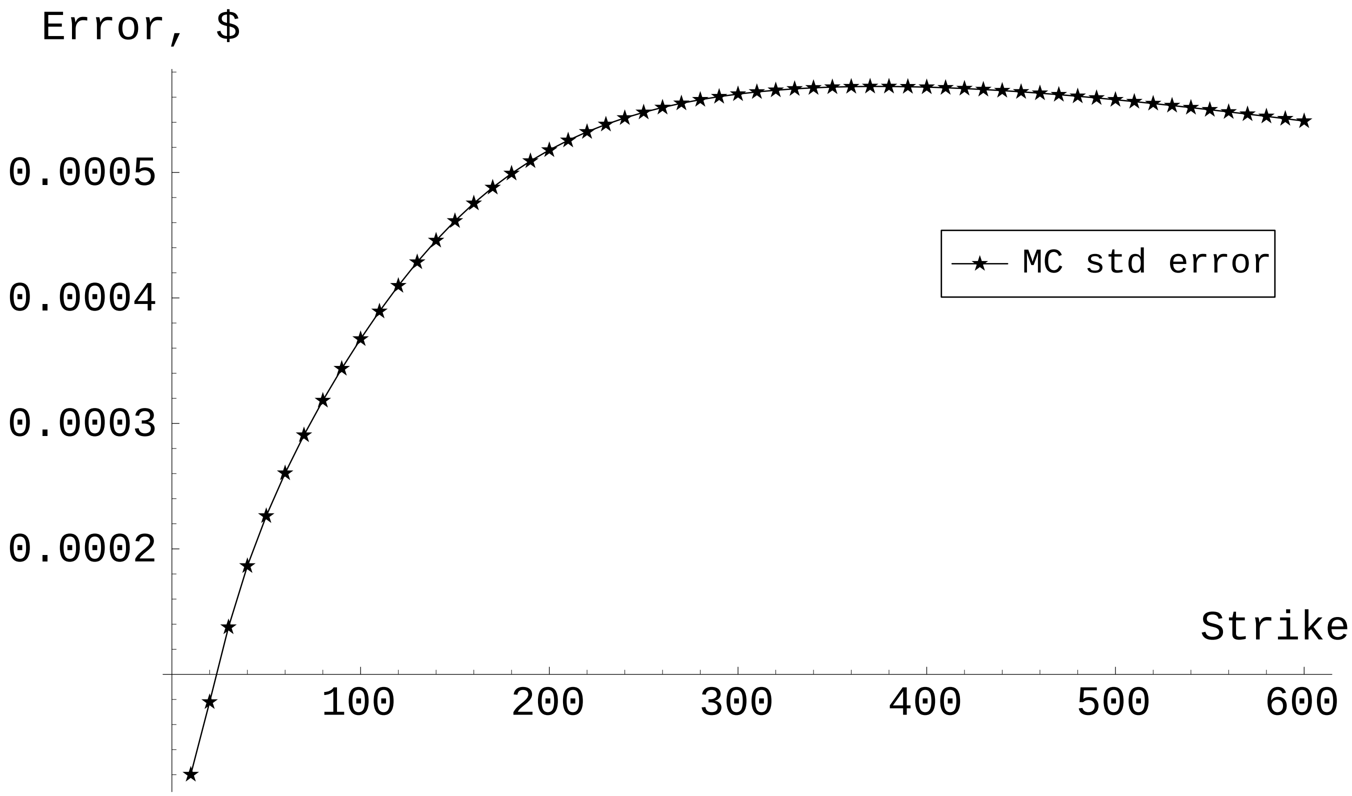
- Five basic classes
  - Monte Carlo simulation
  - Moment matching approaches
  - Curran's method
  - Density perturbation techniques
  - Convolution method
- Drop $_T$ in the following

2.1. Monte Carlo Simulation

- Basic idea: brute force simulation of the (correlated) distributions of S_{t_i}
- Standard approach: generate independent increments using $(0,1]$ RNG and normal transformation
- Enhancements
  - Antithetic variates
  - Control variates
    - Geometric average
    - Vanilla portfolio
  - Quasi-random sequences (Faure, Sobol...)
- Properties
  - Speed: very slow; $\mathcal{T} \sim$ (small const) $\frac{n}{\epsilon^2}$
  - Precision (ϵ): Arbitrarily high given enough time
  - Easy to implement if high precision not required
  - Applicable to any payoff
  - Difficulties computing Greeks

2.1. Monte Carlo Simulation (2)

- Standard error vs. strike:
  - 5 year annual Asian-end call, $S_0 = 100$, $r = 0.06$, $y = 0.02$, $\sigma = 0.50$
  - 100 million Sobol paths, dual optimised control variate



2.2. Moment Matching Approaches

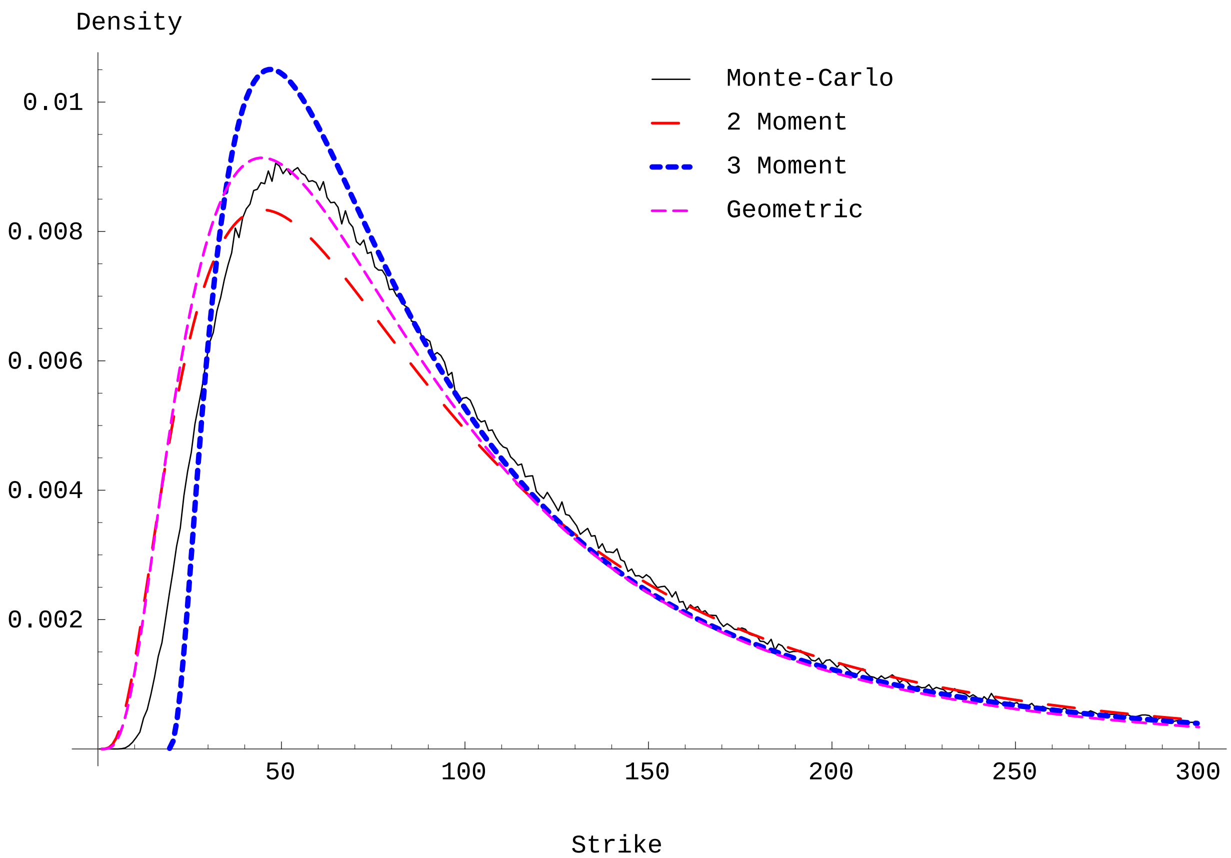
- Oldest and most widely used set of techniques: Ritchken, Sankarasubramaniam, Vijh (1989); Levy (1990)
- Basic idea:
  - Choose a (tractable) family of distributions defined by m parameters
  - Fit the parameters to m moments of the target distribution
- Family almost always based on normal distribution (similarity to geometric)
  - 2 moments: lognormal
    - $\phi\,(a \equiv \ln(A)) \sim n(\tilde{\mu}, \sigma^2)$
    - $e^{\tilde{\mu}+\sigma^2/2} = \mu_1$; $\mu_1^2\Big(e^{\sigma^2} - 1\Big) = \mu_2$
    - Quick implementation, trivial solution for parameters
    - Convenient because Black-Scholes framework is retained
    - Interpretation in terms of effective forward/yield, volatility

2.2. Moment Matching Approaches (2)

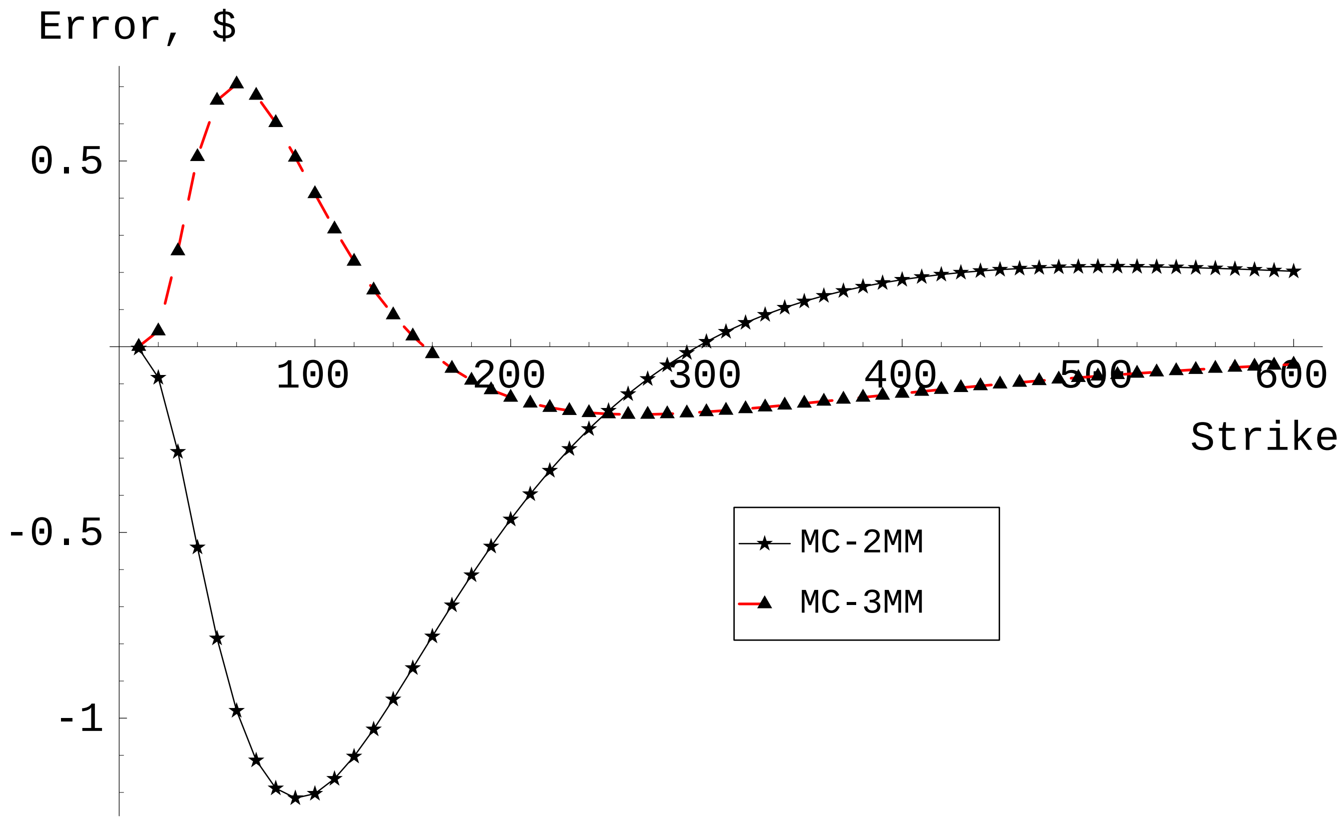
- 3 moments: displaced lognormal (Milevsky and Posner, 1998: also 4MM)
  - $\phi\left(\ln(A-B)\right) \sim n(\tilde{\mu}, \sigma^2)$
  - $B + e^{\tilde{\mu}+\sigma^2/2} = \mu_1;\ e^{2\tilde{\mu}+\sigma^2}\left(e^{\sigma^2}-1\right) = \mu_2;\ e^{3\tilde{\mu}+3\sigma^2/2}\left(e^{3\sigma^2}-3e^{\sigma^2}+2\right) = \mu_3$
  - Solve cubic equation for e^{σ^2}, then work downwards
  - Slight additional complication to Black-Scholes
  - Some artefacts for low strikes
- Advantages of these methods
  - Quick to implement
  - Robust (closed form or near-closed form)
  - Rapid execution for Asians: mth moment can be calculated in $\mathcal{O}(m!\,n)$ time
- Disadvantages
  - Inaccurate for few moments ($m = 2$) and large volatilities/long maturity
  - For higher moments, shape error limits accuracy (non-convergent)
  - Slow execution for baskets: mth moment requires $\mathcal{O}(n^m)$ time
  - ASAE options: joint distribution required; not generally available for $m > 2$

2.2. Moment Matching Approaches (3)

- 5 year annual average density (parameters as above)



- 5 year annual Asian call errors (parameters as above)



2.3. Curran's Method

- Based on moment methods, but distinct enough to merit separate discussion
- Make use of the inequality $A \geq G$, consider valuation of the payoff *conditioned on* G (Curran 1992a, 1992b):

$$
\begin{aligned}
E[A-K]^{+} &= \int_0^\infty dG\,\phi(G) \int_K^\infty dA\,\phi(A|G)\,(A-K) \\
&= \int_0^\infty dG\,\phi(G) \int_{\max(K,G)}^\infty dA\,\phi(A|G)\,(A-K) \\
&= \int_0^K dG\,\phi(G) \int_K^\infty dA\,\phi(A|G)\,(A-K) + \\
&\quad \int_K^\infty dG\,\phi(G)\,\left(E[A|G]-K\right)
\end{aligned}
$$

  - Second term is just a sum of $N(\bullet)$; key observation is that because G and A are highly correlated, this constitutes the bulk of the option's value!
  - Approximations to the first term will have little effect on value. Apply moment methods here (to conditional density)

2.3. Curran's Method (2)

- Depending on approximation to $\int_0^K dG\,\phi(G) \int_K^\infty dA\,\phi(A|G)\,(A-K)$, we can trade-off performance vs. accuracy:

  1) Approximate $\phi(A-G|G)$ as lognormal, with parameters depending on G. Numerical integration of conditioned Black-Scholes formulae is required. Computationally intensive, but highly accurate.

  2) As 1, but choose parameters of $\phi(A-G|G)$ as constants versus G. Numerical integration still required, but much faster.

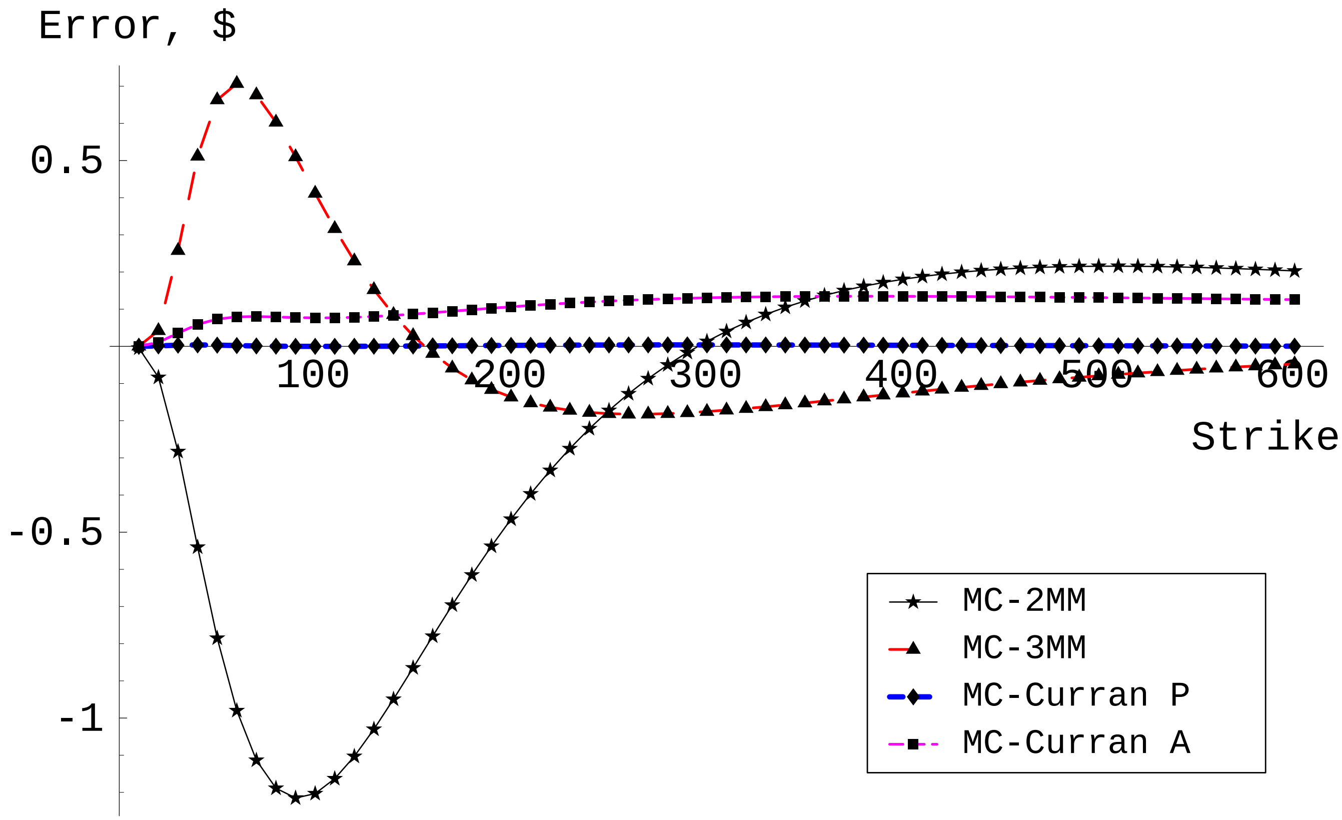
  3) Approximate inner integral $E[(A-K)^+|G]$ by $[E(A|G)-K]^+$ and replace lower integration limit ($G=0$) by $G^* = G : E(A|G) = K$. Numerical integration no longer required (very fast).

  ... many other approximations are possible!

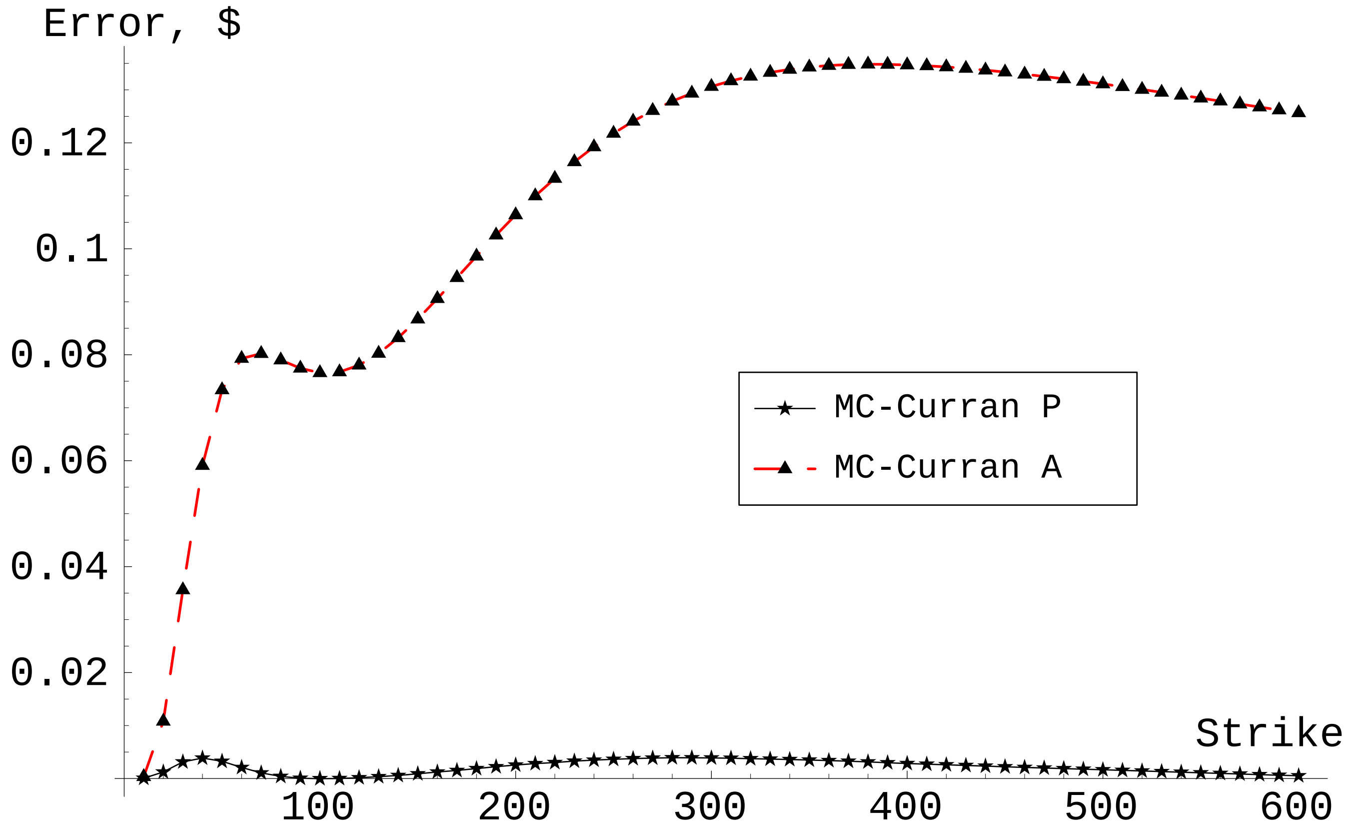
- Fastest approximations are $\mathcal{T} \sim (c_1 + c_2)n$ for Asians and $\mathcal{T} \sim c_1 n + c_2 n^2$ for baskets, with c_1 small and c_2 tiny.
- Slowest approximation follows above scaling but with c_1 and c_2 proportional to number of integration points.
- Theoretically not convergent, but may not matter...
- Method not applicable for ASAE

2.3. Curran's Method (3)

- 5 year annual Asian call errors: Curran vs. Moment matching



- 5 year annual Asian call errors: Comparison of precise versus approximate forms of Curran's method



2.4. Density Perturbation Techniques

- Edgeworth expansion, Gram-Charlier expansion
- Long history in probability theory, finance (Jarrow and Rudd, 1982; Turnbull and Wakeman, 1989).
- Make use of a *reference* distribution $\phi'(B)$, chosen to approximate $\phi(A)$ in some way (i.e, with the same first m moments), but otherwise with no relation between B and A. E.g., B is a lognormal variable with parameters chosen to match the first 2 moments of A.
- Consider characteristic functions of A, B:

$$\tilde{\phi}(k) = E[e^{\iota k A}];\ \tilde{\phi}'(k) = E[e^{\iota k B}]$$

- Basic idea is to express $\phi(k)$ relative to $\phi'(k)$:

$$\tilde{\phi}(k) = \tilde{\phi}'(k)\frac{\tilde{\phi}(k)}{\tilde{\phi}'(k)}$$

  and then expand the ratio of the two characteristic functions in k.

- Employ the cumulant expansion

$$\tilde{\phi}(k) = \exp\left(\iota k \mu_\phi + \frac{(\iota k)^2}{2}\sigma_\phi^2 + \frac{(\iota k)^3}{6}\kappa_{3,\phi} + \frac{(\iota k)^4}{24}\kappa_{4,\phi} + ... + \frac{(\iota k)^m}{m!}\kappa_{m,\phi} + ...\right)$$

2.4. Density Perturbation Techniques (2)

- Then (assuming, for example, that the first two moments are identical):

$$\tilde{\phi}(k) \sim \tilde{\phi}'(k) \left[1 + \frac{(\iota k)^3}{6} (\kappa_{3,\phi} - \kappa_{3,\phi'}) + \frac{(\iota k)^4}{24} (\kappa_{4,\phi} - \kappa_{4,\phi'}) + ... \right]$$

- Using properties of the Fourier transform:

$$\phi(A) \sim \left(1 + \frac{\kappa_{3,\phi} - \kappa_{3,\phi'}}{6} \frac{\partial^3}{\partial A^3} + \frac{\kappa_{4,\phi} - \kappa_{4,\phi'}}{24} \frac{\partial^4}{\partial A^4} + ... \right) \phi'(A)$$

- Hence option values can be computed by:

$$\int_K^\infty dA\, \phi(A)(A-K) \sim \int_K^\infty dA\, \phi'(A)(A-K) + \frac{\kappa_{3,\phi} - \kappa_{3,\phi'}}{6} \left. \frac{\partial \phi'(A)}{\partial A} \right|_{A=K} + \frac{\kappa_{4,\phi} - \kappa_{4,\phi'}}{24} \left. \frac{\partial^2 \phi'(A)}{\partial A^2} \right|_{A=K} + ...$$

- Problem: for a variety of reasons (including the fact that moments of a lognormal grow exponentially with m), this expansion is divergent (asymptotic). This problem is usually dealt with by truncating the series, but accuracy may not be sufficient for large volatilities.

2.4. Density Perturbation Techniques (3)

- Variation on the above approach: consider the density ϕ of $a \equiv \ln(A)$ instead of A itself.
- Again, choose a reference distribution $\phi'(b)$, chosen to approximate $\phi(a)$. E.g., b is a normal variable with parameters chosen so that the first two moments of A are reproduced.
- Return to characteristic function expansion:

$$\tilde{\phi}(k) = \tilde{\phi}'(k) \frac{\tilde{\phi}(k)}{\tilde{\phi}'(k)}$$

and then expand the ratio of the two characteristic functions in k.

- If $\phi'(b) \sim n(\tilde{\mu}_{\phi'}, \sigma^2_{\phi'})$, then:

$$\tilde{\phi}(k) = e^{\iota k \tilde{\mu}_{\phi'} - \frac{k^2}{2}\sigma^2_{\phi'}} E[e^{\iota k (a - \tilde{\mu}_{\phi'}) + \frac{k^2}{2}\sigma^2_{\phi'}}]$$

- Now, must expand the expectation.
- There is "almost" an interpretation as $E[e^{\iota k(a-b)}]$ where b is independent of a. More on that idea later...

2.4. Density Perturbation Techniques (4)

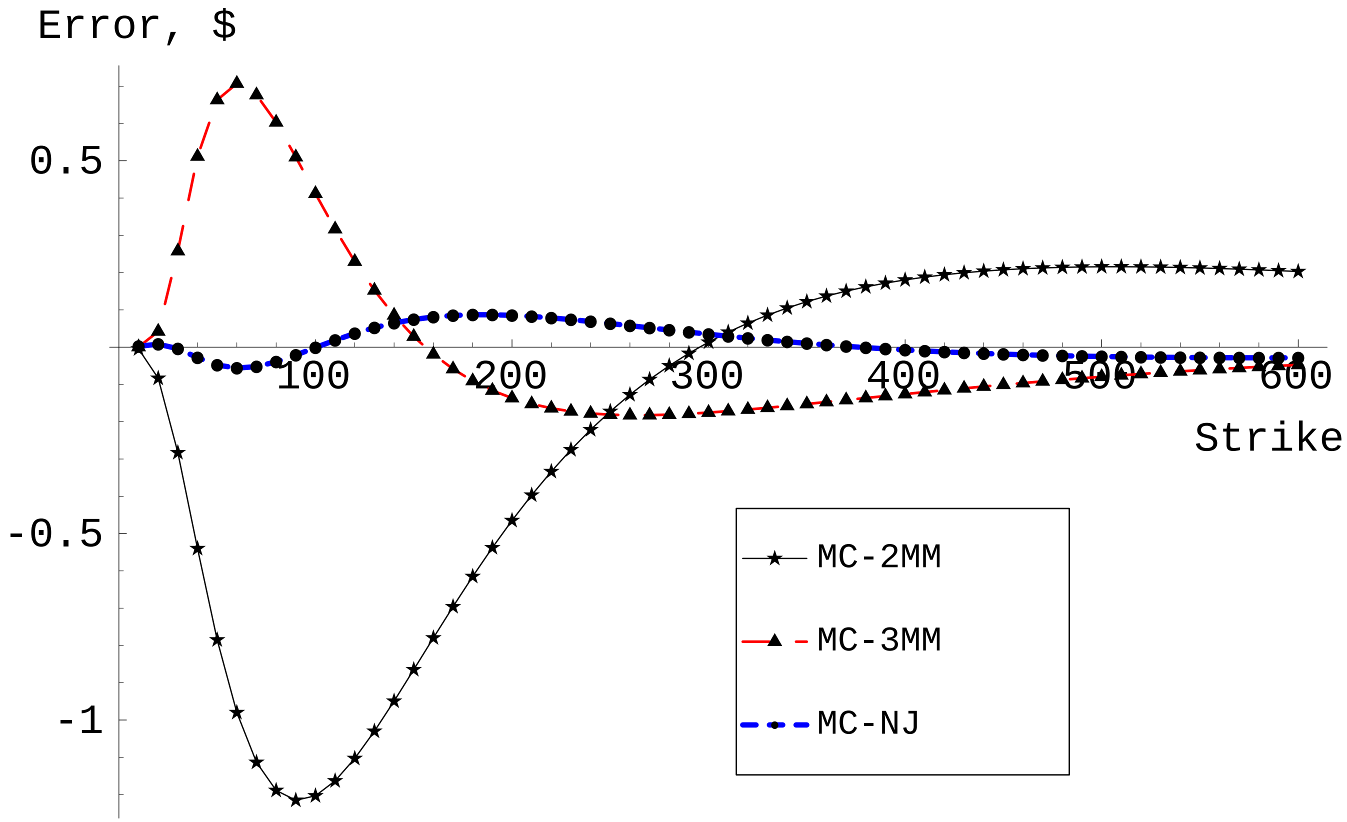
- One new approach (Nengjiu Ju, 2000) expands the expectation in terms of the underlying volatility σ. Coercing the first two moments of b, he finds to $\mathcal{O}(\sigma^6)$:

$$\tilde{\phi}(k) = e^{\iota k \tilde{\mu}_{\phi'} - \frac{k^2}{2}\sigma^2_{\phi'}}(1 - \iota k d_1 - k^2 d_2 + \iota k^3 d_3 + k^4 d_4)$$

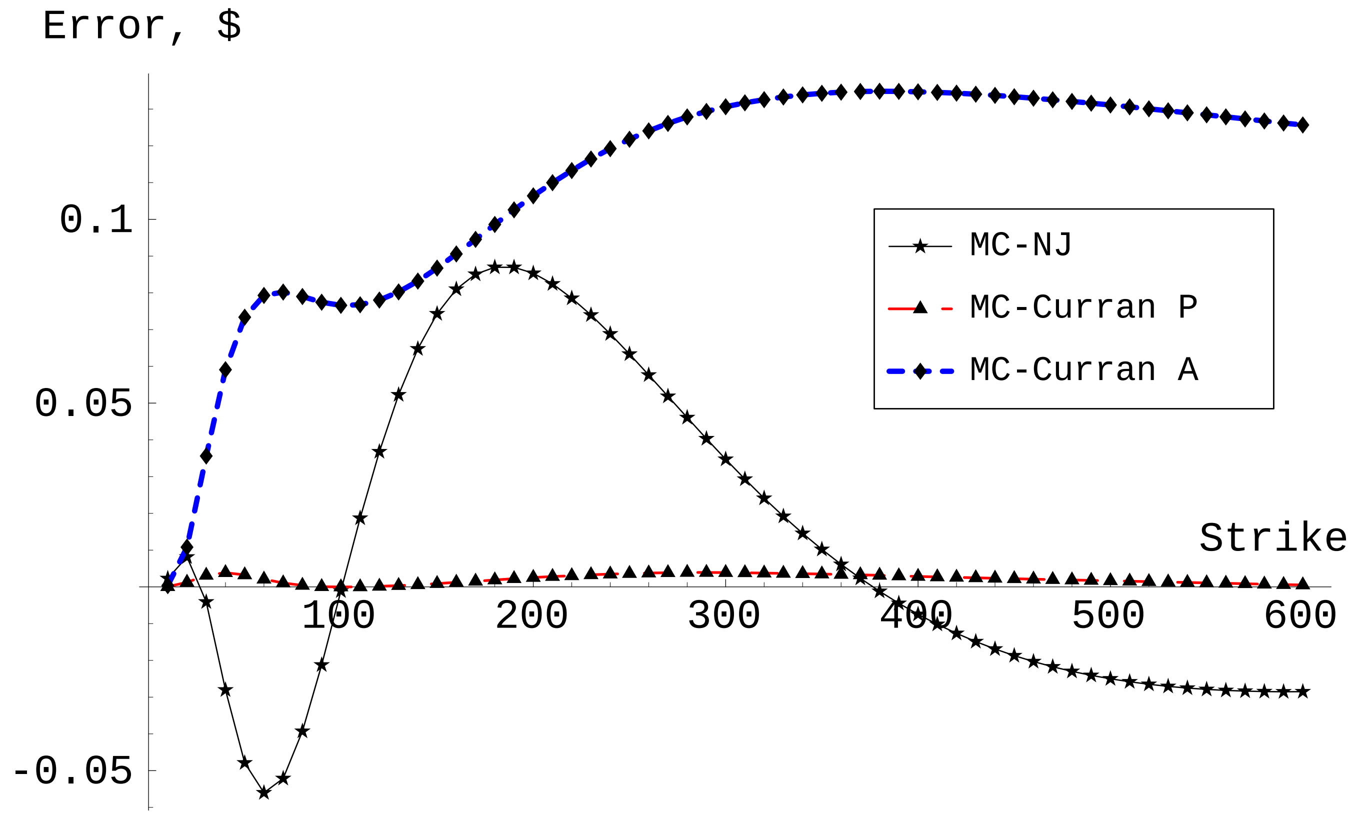
  with d_m simple functions of drifts, volatilities, and time.

- The resulting (quite accurate) valuation formula involves the 2 moment matching approximation plus elementary transcendental corrections similar to those in the Edgeworth expansion.
- For Asians, the method appears $\mathcal{O}(n)$, for baskets $\mathcal{O}(n^3)$
- Method could be extended to ASAE options.

• 5 year annual Asian call errors: Nengjiu Ju vs. Moment matching



- 5 year annual Asian call errors: Nengjiu Ju vs. Curran



2.5. Convolution Method

- The only deterministically convergent technique for Asian options (Carverhill and Clewlow, 1990).
- Make use of the fact that

$$S_1 + S_2 + ... + S_{n-1} + S_n = S_0 R_1(1 + R_2(1 + R_3(1 + ...R_{n-1}(1 + R_n))))$$

  where $R_i \equiv S_i/S_{i-1}$ are independent in Black-Scholes+ setup.
- Working outward from innermost parentheses (backwards in time), we alternatively:
  - Shift (in real space) the distribution of the sum by adding 1 to it
  - Convolve (in Fourier space) the distribution of the sum with the independent distribution of the previous return
- With a little attention paid to smoothness, $\mathcal{T} \sim c_1 \frac{n}{\sqrt{\epsilon}}$ or better is achievable, with c_1 a (large-ish) constant.
- Useless for basket options, not terribly useful for ASAE options because of reliance on independent returns property

3. Characteristic Function Expansions

- What have we learned?
  - There seems to be real power in using "the right" sort of characteristic function expansion around a reference density
  - Methods based on geometric conditioning also seem helpful: we should take advantage of the fact that arithmetic and geometric averages are highly correlated

- Try expansion approach again. Introduce a (normal) reference variable b, but let's actually use it to calculate the characteristic function:

$$E[e^{\iota ka}] = E[e^{\iota k(a-b)+\iota kb}] \equiv E[e^{\iota kv+\iota kb}]$$

  - Define v as above; also use $a = \ln(A) = \ln(\sum_i w_i S_i) \equiv \ln(\sum_i w_i e^{x_i})$ to define x_i (note that we are dropping t subscripts).
  - b could be independent of a (and x_i) as before, or could be correlated with them in some interesting way...
  - Augment x_i by $x_{n+1} \equiv b$.

3. Characteristic Function Expansions (2)

- Now, complete squares to eliminate formal b dependence

$$
\begin{aligned}
E[e^{\iota k(v+b)}] &= \frac{1}{(2\pi)^{\frac{n+1}{2}}|\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx_1 ... dx_{n+1}\, e^{\iota k(v+b) - \frac{1}{2}\sum_{i,j}(x_i - \tilde{\mu}_i)\Sigma_{i,j}^{-1}(x_j - \tilde{\mu}_j)} \\
&= \frac{e^{\iota k \tilde{\mu}_b - \frac{k^2}{2}\sigma_b^2}}{(2\pi)^{\frac{n+1}{2}}|\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx_1 ... dx_{n+1}\, e^{\iota k v - \frac{1}{2}\sum_{i,j}(x_i - \tilde{\mu}'_i)\Sigma_{i,j}^{-1}\left(x_j - \tilde{\mu}'_j\right)} \\
&= E[e^{\iota k b}]E'[e^{\iota k v}],
\end{aligned}
$$

with $\tilde{\mu}_i \equiv E[x_i]$; $\Sigma_{i,j} \equiv \mathrm{cov}(x_i, x_j)$; $\tilde{\mu}'_i \equiv \tilde{\mu}_i + \iota k \Sigma_{i,b} \equiv \tilde{\mu}_i + \iota k\, \mathrm{cov}(x_i, b)$

- The $\tilde{\mu}'_i$ are the means of x_i in a new complex "measure"; the covariances of x_i and x_j are unaffected by this change of measure.
- Clearly, the numerator of the prefactor in the penultimate line of the expectation is just the characteristic function of b, $E[e^{\iota k b}]$.
- This is what we had in mind earlier when we referred to "almost" being able to interpret the Edgeworth/Gram-Charlier approach as an expectation with respect to an independent perturbative b.

3. Characteristic Function Expansions (3)

- Next, change variables to $y_i \equiv x_i - b$ (and drop y_{n+1}):
- In the $'$ measure, the y_i have means $\tilde{\nu}'_i$ and covariances (with y_j) $\Lambda_{i,j}$:

$$\begin{aligned} \tilde{\nu}'_i &\equiv (\tilde{\mu}_i - \tilde{\mu}_b) + \iota k(\Sigma_{i,b} - \Sigma_{b,b}) \equiv \tilde{\nu}_i + \iota k \Lambda_{i,b}, \\ \Lambda_{i,j} &\equiv \Sigma_{i,j} - \Sigma_{i,b} - \Sigma_{j,b} + \Sigma_{b,b} \end{aligned}$$

- We're now in a better position to proceed. The problem of computing the characteristic function of a has been reduced to evaluating:

$$E'[e^{\iota k v}] = E'\left[e^{\iota k \log(\sum_i w_i e^{y_i})}\right].$$

- Define:

$$f \equiv \sum_i w_i e^{y_i} - 1.$$

- f is in some sense small (scales as σ_b^2). Then our expectation becomes:

$$E'[e^{\iota k v}] = E'\left[\exp\left(\iota k \log(1+f)\right)\right] = E'\left[(1+f)^{\iota k}\right]$$

3. Characteristic Function Expansions (4)

- Use the generalised binomial formula to write:

$$
\begin{aligned}
E'[e^{\iota k v}] &= \sum_{m=0}^{\infty} E'\left[\binom{\iota k}{m} f^m\right] \\
&= 1 + \iota k E'[f] + \frac{\iota k(\iota k-1)}{2} E'\left[f^2\right] + \ldots \\
&= 1 + \iota k E'[\textstyle\sum_i w_i e^{y_i} - 1] + \frac{\iota k(\iota k-1)}{2} E'\left[\sum_{i,j} w_i w_j (e^{y_i} - 1)(e^{y_j} - 1)\right] + \ldots \\
&= 1 + \iota k E'[\textstyle\sum_i w_i e^{y_i} - 1] + \frac{\iota k(\iota k-1)}{2} E'\left[\sum_{i,j} w_i w_j e^{y_i + y_j} - 2\sum_i w_i e^{y_i} + 1\right] \\
&\quad + \ldots \\
&= 1 + \iota k\left[\sum_i w_i e^{\nu'_i + \frac{\Lambda_{i,i}}{2}} - 1\right] \\
&\quad + \frac{\iota k(\iota k-1)}{2}\left[\sum_{i,j} w_i w_j e^{\nu'_i + \nu'_j + \frac{\Lambda_{i,i} + \Lambda_{j,j}}{2} + \Lambda_{i,j}} - 2\sum_i w_i e^{\nu'_i + \frac{\Lambda_{i,i}}{2}} + 1\right] + \ldots
\end{aligned}
$$

- Now, notice a few things:
  - ν'_i contain linear terms in ιk; these *shift* the reference distribution in such a way as to give us the correct expectation of each exponential term
  - Truncating the expansion at some m guarantees that moments of A up to and including m will be represented exactly. Alternatively, all higher terms yield no contribution to the moment of order m.

3. Characteristic Function Expansions (5)

- We can think of this as the "natural" expansion method for a desired density around a (correlated) normal density
- How to choose b?
  - Geometric average (numerical results to follow), but first two moments won't be exact until $m = 2$ terms are included
  - Geometric average + an independent piece to give two correct moments from the start...
- What if we rearrange this f expansion into an $\mathcal{E} \equiv \sum_i w_i e^{y_i}$ expansion?
  - Form of results depends on order of truncation in f expansion
  - $\mathcal{O}(f^0)$: $\tilde{\phi} = E[e^{\iota k b}]$
  - $\mathcal{O}(f^1)$: $\tilde{\phi} = E[e^{\iota k b}][(1 - \iota k) + \iota k E'(\mathcal{E}^1)]$
  - $\mathcal{O}(f^2)$: $\tilde{\phi} = E[e^{\iota k b}] \left[\frac{(1-\iota k)(2-\iota k)}{2} + \iota k(2 - \iota k)E'(\mathcal{E}^1) + \frac{\iota k(\iota k-1)}{2} E'(\mathcal{E}^2)\right]$
  - Coefficients generalise to Gamma functions of ιk.

3. Characteristic Function Expansions (6)

- At each order of truncation, the expansion leads to an intuitive approximation to the expected payoff
  - $\mathcal{O}(f^0)$: $E[(e^a - e^k)^+] = E[e^b]N[d_+] - e^k N[d_-]$
  - $\mathcal{O}(f^1)$: $E[(e^a - e^k)^+] = \sum_i w_i E[e^{x_i}]N[d_{+,i}] - e^k N[d_-]$
  - $\mathcal{O}(f^2)$: $E[(e^a - e^k)^+] =$ $\sum_i w_i E[e^{x_i}]N[d_{+,i}] - e^k N[d_-] + \frac{e^k}{2}\sum_{i,j} w_i w_j [n(d_-) - 2n(d_{-,i}) + n(d_{-,i,j})]$

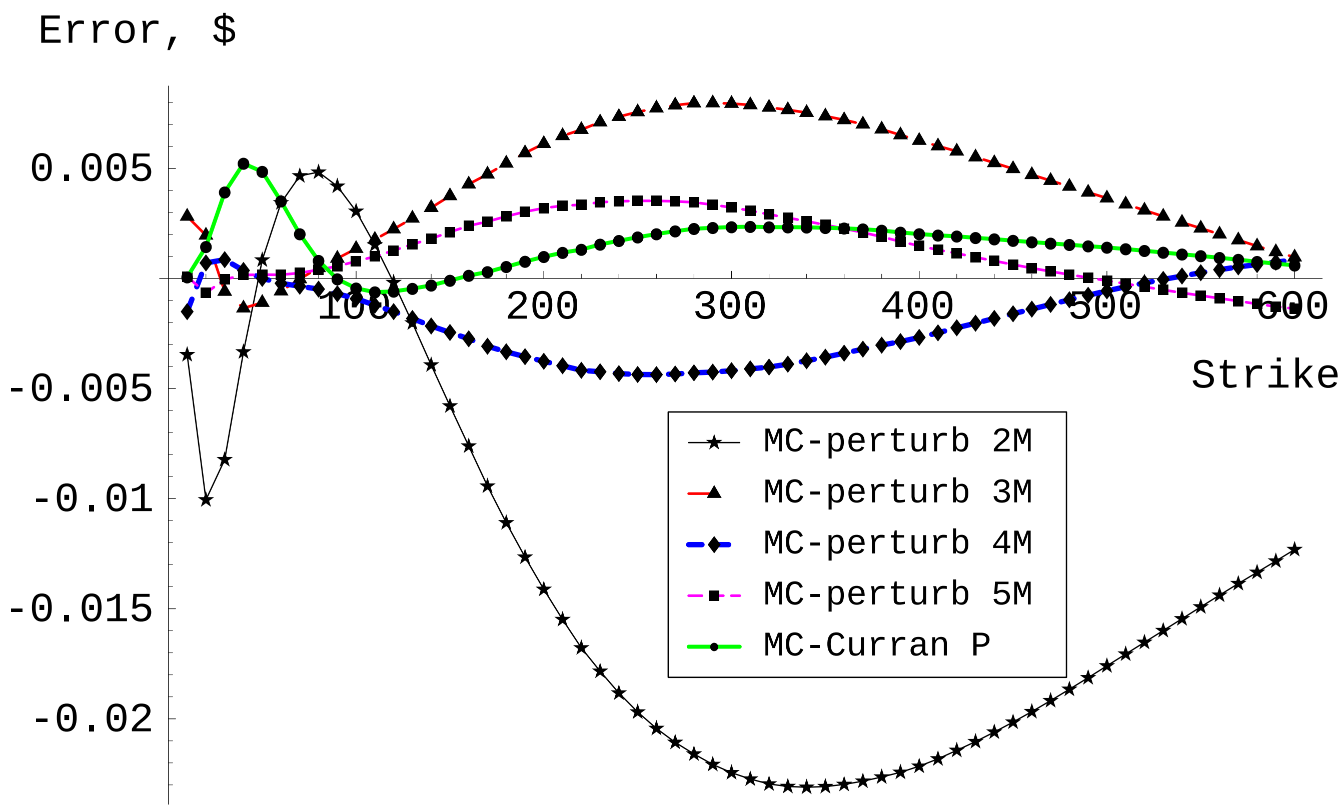
- Number of $N[\bullet]$ is $\mathcal{O}(n)$, but number of $n(\bullet)$ is $\mathcal{O}(n^m)$ for truncation at $\mathcal{O}(f^m)$.
  - This is independent of whether the option is Asian or a basket.
  - For Asians, the normal density terms expand naturally in σ a la Ju, reducing dimensionality...

- Extension to ASAE is straightforward (multi-variate approximation). For fixed units, results are similar to those above; for fixed notional, convexity corrections appear.

- Application to A_p averages is also straightforward; indeed, we first derived the approach in that context.

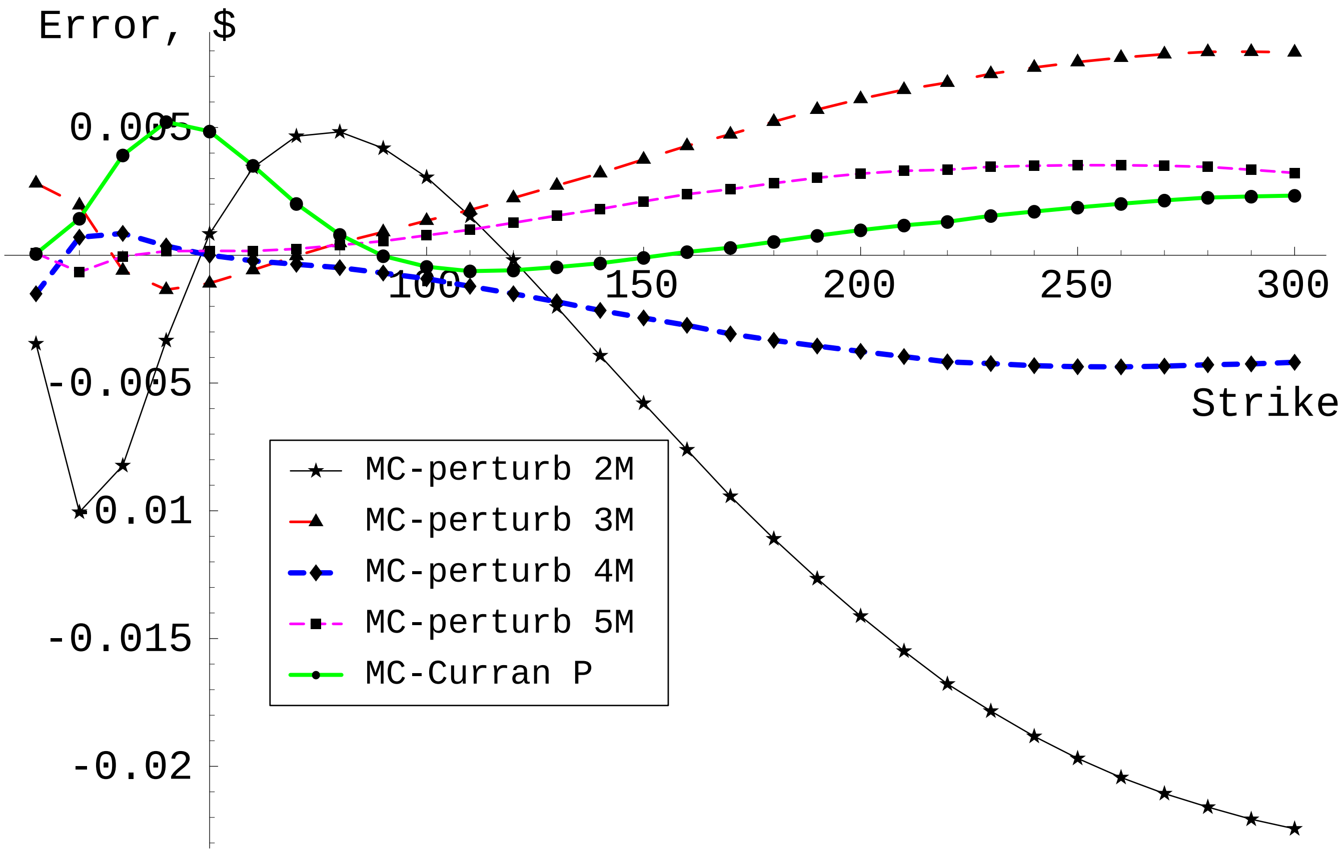
4. Numerical Convergence Properties

- 5 year annual Asian call errors: Perturbation vs. precise Curran



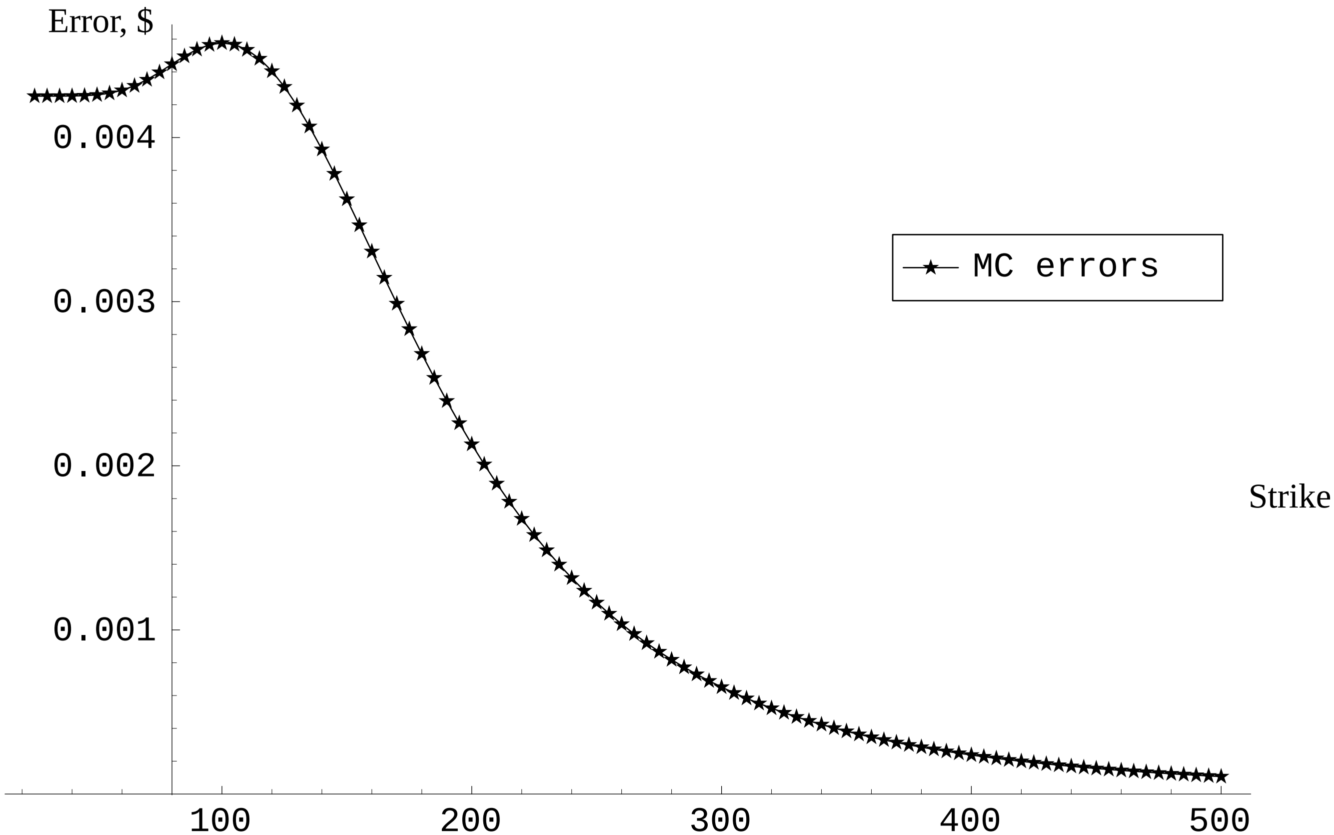
4. Numerical Convergence Properties (2)

- 5 year annual Asian call errors: Perturbation vs. precise Curran



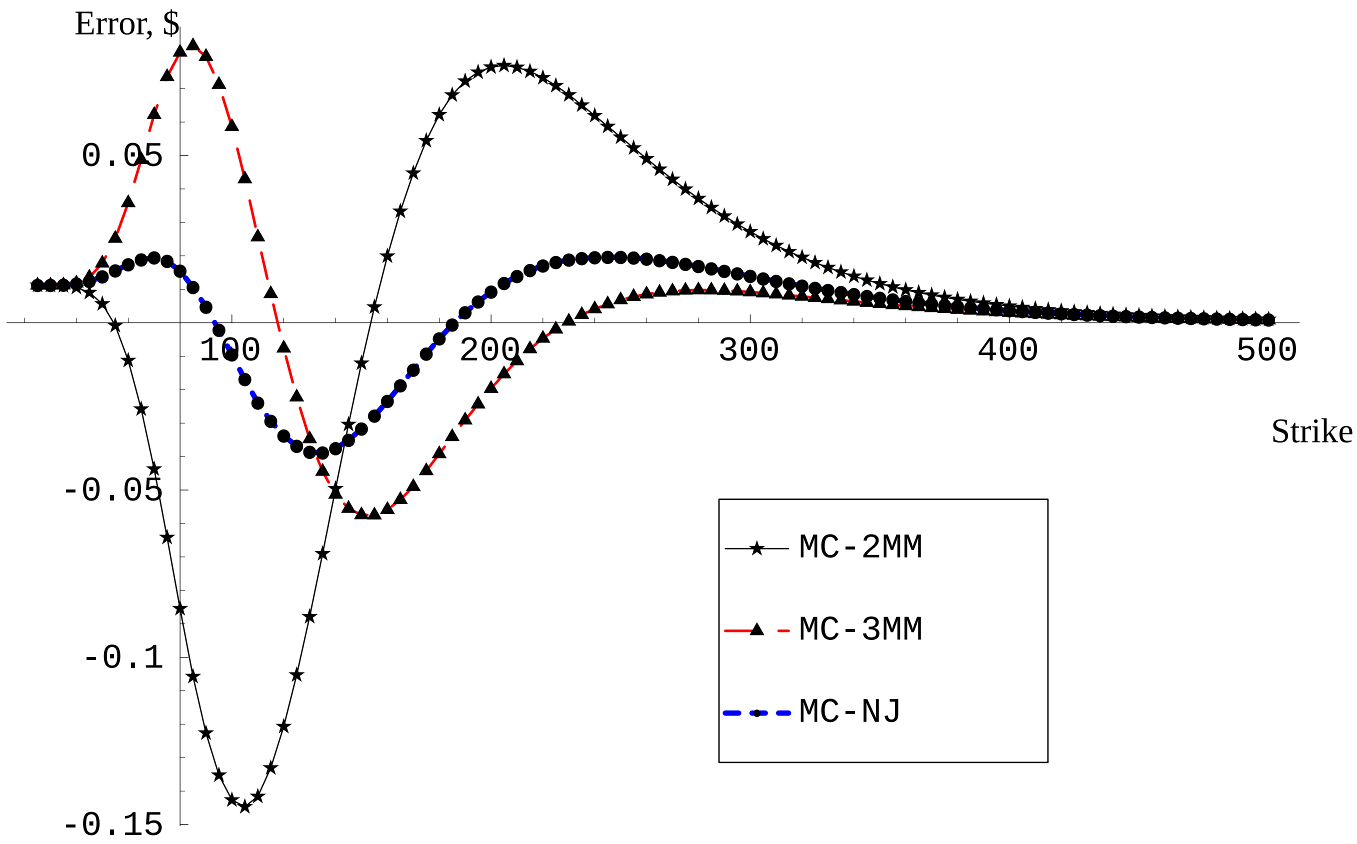
4. Numerical Convergence Properties (3)

- 5 year, 5 component equally-weighted basket call ($\sigma_i = 0.30$, $\rho_{i,j} = 0.0$, other parameters as before): Monte Carlo standard error vs. strike



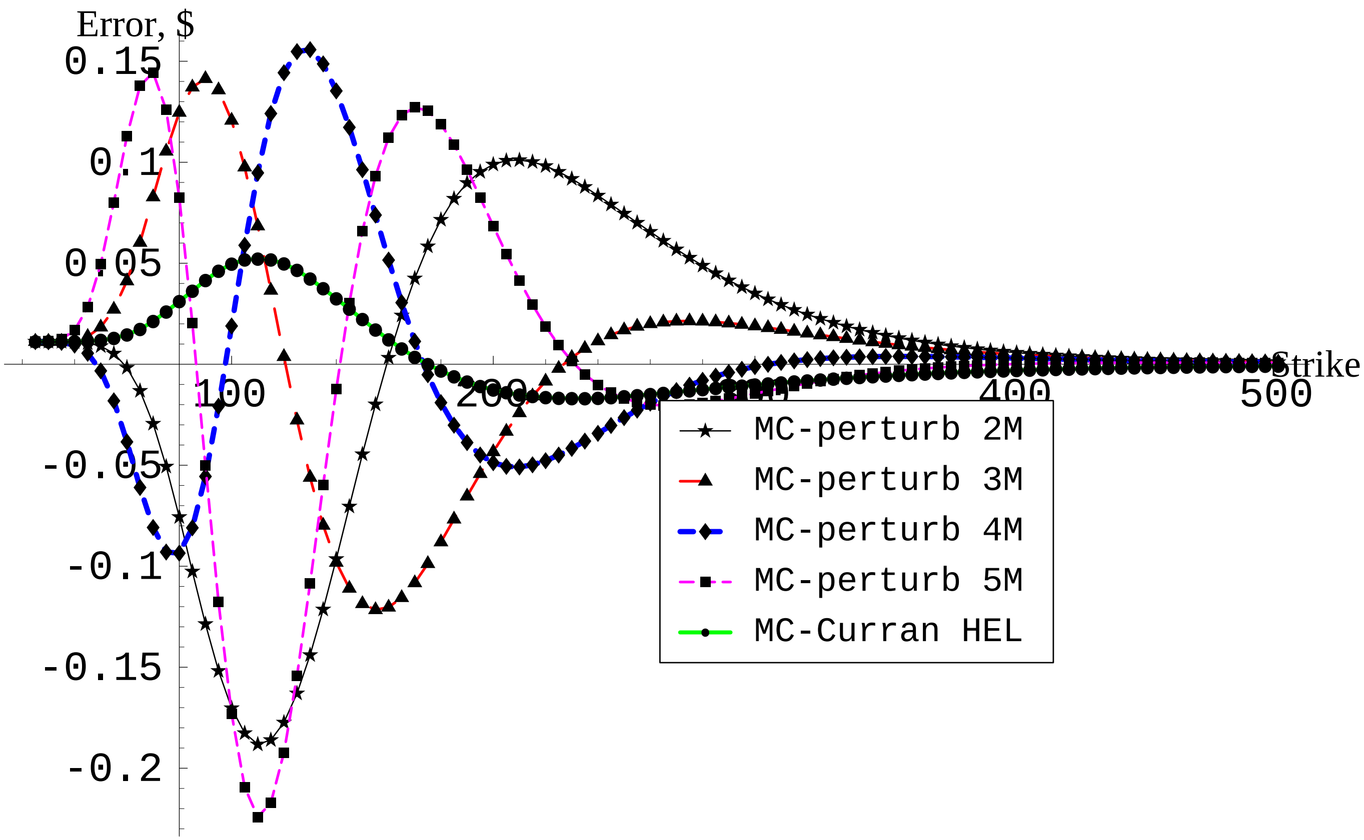
4. Numerical Convergence Properties (4)

- 5 year basket call errors: Nengjiu Ju vs. Moment matching



4. Numerical Convergence Properties (5)

- 5 year basket call errors: Perturbation vs. precise Curran



5. Theoretical Convergence Properties

- Numerical convergence properties look quite good for Asians, even for fairly extreme parameters.
- For baskets, especially with high volatilities/low correlations, preliminary numerical results suggest convergence is much weaker.
- Problem: assumption that $(1+f)^{\iota k}$ can be expanded in k. Especially for $\sigma_b \geq 1$, it is far from clear that the series (or its expectation) is convergent.
- We're not sure whether the method has a finite radius of convergence or whether it's simply asymptotic.
- We believe that similar issues may apply to Ju's method.

6. Conclusions

- A new method for expanding the characteristic function of arithmetic (and other) averages has been derived.
- While the approach is most closely related to earlier characteristic function expansion techniques, much of its novelty lies in the use of a reference distribution correlated to the components of the average.
- Since the geometric average is a natural candidate for the reference distribution, our method also ties in nicely with that segment of the Asian option literature that exploits the close relationship between geometric and arithmetic averages.
- The approach leads to intuitive corrections to the characteristic function and option values. In particular, excellent control over moments is provided.
- Numerical results are good for Asian options, less so for basket options. Theoretical implications for this (and related) methods need more analysis.