

Asymptotic behavior of distribution densities in stochastic volatility models

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Seminar talk at Stanford University

09/28/2007

Samuelson's model of the stock price

In their celebrated work on pricing of options, Black and Scholes used Samuelson model of the stock price. Samuelson suggested to describe the random behavior of the stock price by a diffusion process X_t satisfying the following stochastic differential equation:

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x_0,$$

where μ is a real constant, σ is a positive constant, and W_t is a standard Brownian motion. The constants $\mu \in \mathbb{R}^1$ and $\sigma > 0$ are called the drift and the volatility of the stock, respectively.

Explicit formula

The following formula holds for the stock price process in Samuelson's model:

$$X_t = x_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}.$$

The process X_t is called a geometric Brownian motion.

Stock price distribution density

The *distribution function* of the stock price X_t is given by

$$r \mapsto \mathbb{P}(X_t < r), \quad 0 \leq r < \infty,$$

where \mathbb{P} stands for the Wiener measure.

The *distribution density* of X_t (if it exists) is a function D_t on $[0, \infty)$ such that

$$\mathbb{P}(X_t < r) = \int_0^r D_t(u) du$$

for all $r > 0$.

Stock price distribution in Samuelson's model

The distribution density of the stock price process X_t in Samuelson's model can be computed explicitly. We have

$$D_t(x) = \frac{1}{\sqrt{2\pi t\sigma x}} \exp \left\{ -\frac{\left(\log \frac{x}{x_0} - \left(\mu - \frac{1}{2}\sigma^2 \right) t \right)^2}{2t\sigma^2} \right\}.$$

Such densities are called log-normal.

Black-Scholes model

The Black-Scholes formula for the price of a European call option at $t = 0$ is the following:

$$C_{BS} = x_0 N(d_1) - Ke^{-rT} N(d_2),$$

where

$$d_1 = \frac{\log x_0 - \log K + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left\{-\frac{y^2}{2}\right\} dy.$$

Here T stands for the expiration date, K is the strike price, and r denotes the interest rate.

Constant volatility assumption is erroneous

It is assumed in the Black-Scholes model that the volatility is constant. However, it has been observed that the implied volatility for a European call option depends on the strike price and the expiration date. This fact contradicts the constant volatility assumption. The implied volatility considered as a function of the strike price is locally convex near its minimum. This is the so-called “volatility smile” effect.

Stochastic volatility models

To improve the Black-Scholes model, various stochastic volatility models were suggested. Some of the well-known stochastic volatility models are the Hull-White model where the volatility is a geometric Brownian motion; the Stein-Stein model where the absolute value of an Ornstein-Uhlenbeck process is used as the volatility process; and the Heston model where the volatility is a Cox-Ingersoll-Ross process.

Let us first consider the following general stochastic volatility model:

$$\begin{cases} dX_t = \mu X_t dt + f(Y_t) X_t dW_t \\ dY_t = b(t, Y_t) dt + \sigma(t, Y_t) dZ_t \end{cases}$$

Here, $\mu \in \mathbb{R}^1$; b and σ are continuous functions on $[0, T] \times \mathbb{R}^1$; W_t and Z_t are independent one-dimensional

standard Brownian motions; and f is a nonnegative function on \mathbb{R}^1 . The process X_t plays the role of the stock price process, while $f(Y_t)$ is the volatility process. The initial conditions for the processes X_t and Y_t will be denoted by x_0 and y_0 , respectively. We also assume that the second equation in the model above has a unique strong solution Y_t .

Explicit formula

Under appropriate conditions,

$$X_t = x_0 \exp \left\{ \mu t - \frac{1}{2} \int_0^t f(Y_s)^2 ds + \int_0^t f(Y_s) dW_s \right\}.$$

Distribution densities

We will denote by D_t the distribution density of the stock price X_t , and by m_t the mixing distribution density associated with the volatility process $f(Y_t)$, that is, the distribution density of the random variable

$$\alpha_t = \left\{ \frac{1}{t} \int_0^t f(Y_s)^2 ds \right\}^{\frac{1}{2}}.$$

The Hull-White model

The stock price process X_t and the volatility process Y_t in the Hull-White model satisfy the following system of stochastic differential equations:

$$\begin{cases} dX_t = \mu X_t dt + Y_t X_t dW_t \\ dY_t = \nu Y_t dt + \xi Y_t dZ_t. \end{cases}$$

The Stein-Stein model

In the Stein-Stein model, the stock price process X_t and the volatility process Y_t satisfy the following system of stochastic differential equations:

$$\begin{cases} dX_t = \mu X_t dt + |Y_t| X_t dW_t \\ dY_t = q(m - Y_t) dt + \sigma dZ_t. \end{cases}$$

We will only consider the case where $\mu \in \mathbb{R}$, $q \geq 0$, $m = 0$, and $\sigma > 0$. Then the Stein-Stein model becomes

$$\begin{cases} dX_t = \mu X_t dt + |Y_t| X_t dW_t \\ dY_t = -q Y_t dt + \sigma dZ_t. \end{cases}$$

The solution to the second stochastic differential equation in (??) is an Ornstein-Uhlenbeck process, for which the

long run mean equals zero. The reason why we restrict ourselves to the case where $m = 0$ is that CIR-processes and Ornstein-Uhlenbeck processes with $m = 0$ can be dealt with in a similar way, using Bessel processes. The case where $m \neq 0$ is more complicated, and new methods may be needed.

The Heston model

In the Heston model, the stock price process X_t and the volatility process Y_t satisfy the following system of stochastic differential equations:

$$\begin{cases} dX_t = \mu X_t dt + \sqrt{Y_t} X_t dW_t \\ dY_t = (a + bY_t) dt + c\sqrt{Y_t} dZ_t. \end{cases} \quad (1)$$

We will consider the case where $\mu \in \mathbb{R}$, $a \geq 0$, $b \leq 0$, and $c > 0$. It is often assumed that the volatility equation in (1) is written in a mean-reverting form. Then the Heston model becomes

$$\begin{cases} dX_t = \mu X_t dt + \sqrt{Y_t} X_t dW_t \\ dY_t = r(m - Y_t) dt + c\sqrt{Y_t} dZ_t, \end{cases}$$

where $r \geq 0$, $m \geq 0$, and $c > 0$.

Explicit formula

The following formula holds for the stock price process:

$$X_t = x_0 \exp \left\{ \mu t - \frac{1}{2} \int_0^t Y_s^2 ds + \int_0^t Y_s dW_s \right\}.$$

Integral representation

The following equality holds:

$$D_t(x) = \frac{1}{x_0 e^{\mu t}} \int_0^\infty L \left(t, y, \frac{x}{x_0 e^{\mu t}} \right) m_t(y) dy,$$

where L is the log-normal density defined by

$$L(t, y, v) = \frac{1}{\sqrt{2\pi t y v}} \exp \left\{ -\frac{\left(\log v + \frac{t y^2}{2} \right)^2}{2 t y^2} \right\}.$$

It follows that

$$D_t(x_0 e^{\mu t} x) = \frac{1}{x_0 e^{\mu t} \sqrt{2\pi t}} x^{-\frac{3}{2}} \int_0^\infty y^{-1} m_t(y) \exp \left\{ -\left[\frac{1}{2 t y^2} \log^2 x + \frac{t y^2}{8} \right] \right\} dy.$$

Symmetry of the stock price distribution density

$$D_t(x_0 e^{\mu t} x) = x^{-3} D_t\left(\frac{1}{x}; 0, \nu, \xi, 1, y_0\right).$$

Asymptotic behavior of the mixing distribution

The Hull-White model. Special case:

$$\begin{aligned} m_t\left(y; \frac{1}{2}, 1, 1\right) \\ = c_1 y^{c_2} (\log y)^{c_3} \exp\left\{-\frac{1}{2t\xi^2} \left(\log \frac{y}{y_0} + \frac{1}{2} \log \log \frac{y}{y_0}\right)^2\right\} \\ \left(1 + O\left((\log y)^{-\frac{1}{2}}\right)\right), \quad y \rightarrow \infty. \end{aligned}$$

Formulas for the constants

$$c_1 = \frac{1}{\sqrt{\pi t}} 2^{-\frac{1}{2t}} \exp\left\{-\frac{(\log 2)^2}{2t}\right\}, \quad c_2 = -1 - \frac{1 - 2 \log 2}{2t},$$

and

$$c_3 = -\frac{1 + 2 \log 2}{4t}.$$

The law of the time integral

$\tilde{m}_T^{(0)}$ denotes the law of the time integral

$$A_T^{(\rho)} = \int_0^T \exp \{2(\rho u + Z_u)\} du$$

of a geometric Brownian motion.

The following formula holds:

$$\begin{aligned} \tilde{m}_T^{(0)}(y) &= c_1 2^{-1-c_3} T^{-\frac{c_2+1}{2}} y^{\frac{c_2-1}{2}} \left(\log \frac{y}{T}\right)^{c_3} \\ &\exp \left\{ -\frac{1}{2T} \left(\log \sqrt{\frac{y}{T}} + \frac{1}{2} \log \log \sqrt{\frac{y}{T}} \right)^2 \right\} \\ &\left(1 + O \left((\log y)^{-\frac{1}{2}} \right) \right), \quad y \rightarrow \infty, \end{aligned}$$

where c_1 , c_2 , and c_3 are as above with t replaced by T .

The Hull-White model. General case:

$$\begin{aligned}
& m_t(y; \nu, \xi, y_0) \\
&= c_1 y^{c_2} (\log y)^{c_3} \exp \left\{ -\frac{1}{2t\xi^2} \left(\log \frac{y}{y_0} + \frac{1}{2} \log \log \frac{y}{y_0} \right)^2 \right\} \\
&\quad \left(1 + O \left((\log y)^{-\frac{1}{2}} \right) \right)
\end{aligned}$$

as $y \rightarrow \infty$.

Formulas for the constants:

$$c_1 = \frac{1}{\xi \sqrt{\pi t}} 2^{-\frac{1}{2t\xi^2}} y_0^{\frac{2\log 2 - 1}{2t\xi^2} - \alpha} \exp \left\{ -\frac{(\log 2)^2}{2t\xi^2} \right\} \exp \left\{ -\frac{\alpha^2 \xi^2 t}{2} \right\},$$

$$c_2 = \alpha - 1 + \frac{1 - 2 \log 2}{2t\xi^2},$$

$$c_3 = \frac{\alpha}{2} - \frac{1 + 2 \log 2}{4t\xi^2},$$

and

$$\alpha = \frac{2\nu - \xi^2}{2\xi^2}.$$

The law of the time integral

The following formula holds for the law of the time inte-

graph of a geometric Brownian motion:

$$\begin{aligned} \tilde{m}_T^{(\rho)}(y) &= C_1 2^{-1-C_3} T^{-\frac{C_2+1}{2}} y^{\frac{C_2-1}{2}} \left(\log \left(\frac{y}{T} \right) \right)^{C_3} \\ &\quad \exp \left\{ -\frac{1}{2T} \left(\log \sqrt{\frac{y}{T}} + \frac{1}{2} \log \log \sqrt{\frac{y}{T}} \right)^2 \right\} \\ &\quad \left(1 + O \left((\log y)^{-\frac{1}{2}} \right) \right) \end{aligned}$$

as $y \rightarrow \infty$.

Formulas for the constants:

$$\begin{aligned} C_1 &= \frac{1}{\sqrt{\pi T}} 2^{-\frac{1}{2T}} \exp \left\{ -\frac{(\log 2)^2}{2T} \right\} \exp \left\{ -\frac{\rho^2 T}{2} \right\}, \\ C_2 &= \rho - 1 + \frac{1 - 2 \log 2}{2T}, \quad \text{and} \quad C_3 = \frac{\rho}{2} - \frac{1 + 2 \log 2}{4T}. \end{aligned}$$

Asymptotic behavior of the mixing distribution

The Hull-White model. Case where $x \rightarrow 0$. Let $-\infty < \nu < \infty$, $\xi > 0$, $y_0 > 0$, and $t > 0$. Then there exists a positive constant $b = b(\nu, \xi, y_0, t)$ such that

$$m_t(y; \nu, \xi, y_0) = b y^{2\alpha-1} \exp \left\{ -\frac{y_0^2}{2t\xi^2 y^2} \right\} (1 + O(y^2))$$

as $y \rightarrow 0$.

Asymptotic behavior of the stock price density

Asymptotics near infinity. The Hull-White model.
Special case:

$$\begin{aligned}
 D_t \left(x; 0, \frac{1}{2}, 1, 1, 1 \right) &= c_1 2^{\frac{c_2-1-2c_3}{2}} t^{-\frac{c_2+1}{2}} x^{-2} \\
 &(\log x)^{\frac{c_2-1}{2}} \left(\log \left[\frac{2 \log x}{t} \right] \right)^{c_3} \\
 &\exp \left\{ -\frac{1}{2t} \left(\log \sqrt{\frac{2 \log x}{t}} + \frac{1}{2} \log \log \sqrt{\frac{2 \log x}{t}} \right)^2 \right\} \\
 &\left(1 + O \left((\log \log x)^{-\frac{1}{2}} \right) \right)
 \end{aligned}$$

as $x \rightarrow \infty$, where the constants c_1 , c_2 , and c_3 are as above.

Asymptotics near infinity and zero. The Hull-White model. General case:

Let $-\infty < \mu < \infty$, $-\infty < \nu < \infty$, $\xi > 0$, $x_0 > 0$, $y_0 > 0$, and $t > 0$. Then

$$\begin{aligned}
D_t(x_0 e^{\mu t} x; \mu, \nu, \xi, x_0, y_0) &= \frac{c_1}{x_0 e^{\mu t}} 2^{\frac{c_2-1-2c_3}{2}} t^{-\frac{c_2+1}{2}} x^{-2} \\
&\quad (\log x)^{\frac{c_2-1}{2}} \left(\log \left[\frac{2 \log x}{t} \right] \right)^{c_3} \\
&\quad \exp \left\{ -\frac{1}{2t\xi^2} \left(\log \left[\frac{1}{y_0} \sqrt{\frac{2 \log x}{t}} \right] + \frac{1}{2} \log \log \left[\frac{1}{y_0} \sqrt{\frac{2 \log x}{t}} \right] \right)^2 \right\} \\
&\quad \left(1 + O \left((\log \log x)^{-\frac{1}{2}} \right) \right)
\end{aligned}$$

as $x \rightarrow \infty$, and

$$\begin{aligned}
D_t(x_0 e^{\mu t} x; \mu, \nu, \xi, x_0, y_0) &= \frac{c_1}{x_0 e^{\mu t}} 2^{\frac{c_2-1-2c_3}{2}} t^{-\frac{c_2+1}{2}} x^{-1} \\
&\quad \left(\log \frac{1}{x} \right)^{\frac{c_2-1}{2}} \left(\log \left[\frac{2 \log \frac{1}{x}}{t} \right] \right)^{c_3} \\
&\quad \exp \left\{ -\frac{1}{2t\xi^2} \left(\log \left[\frac{1}{y_0} \sqrt{\frac{2 \log \frac{1}{x}}{t}} \right] + \frac{1}{2} \log \log \left[\frac{1}{y_0} \sqrt{\frac{2 \log \frac{1}{x}}{t}} \right] \right)^2 \right\} \\
&\quad \left(1 + O \left(\left(\log \log \frac{1}{x} \right)^{-\frac{1}{2}} \right) \right)
\end{aligned}$$

as $x \rightarrow 0$.

Asymptotic behavior of the mixing distribution

The Heston model. General case:

For $a \geq 0$, $b \leq 0$, and $c > 0$, there exist $A > 0$, $B > 0$, and $C > 0$ such that

$$m_t(y; a, b, c, y_0) = Ay^{-\frac{1}{2} + \frac{2a}{c^2}} e^{By} e^{-Cy^2} \left(1 + O\left(y^{-\frac{1}{2}}\right)\right)$$

as $y \rightarrow \infty$.

Asymptotic behavior of the mixing distribution

The Heston model. Special case:

The following formula holds:

$$m_t(y, a, 0, c, y_0) = Ay^{-\frac{1}{2} + \frac{2a}{c^2}} e^{By} e^{-Cy^2} \left(1 + O\left(y^{-\frac{1}{2}}\right)\right)$$

as $y \rightarrow \infty$.

Formulas for the constants

$$A = \frac{2^{\frac{1}{4} + \frac{a}{c^2}} y_0^{\frac{1}{4} - \frac{a}{c^2}}}{c\sqrt{t}} \exp\left\{\frac{4y_0}{c^2 t}\right\},$$

$$B = \frac{2\sqrt{2y_0}\pi}{c^2 t}, \quad \text{and} \quad C = \frac{\pi^2}{2c^2 t}.$$

Asymptotic behavior of the stock price distribution

The Heston model.

For $a \geq 0$, $b \leq 0$, and $c > 0$, there exist $A_1 > 0$, $A_2 > 0$, and $A_3 > 0$ such that for every δ with $0 < \delta < \frac{1}{2}$,

$$D_t(x_0 e^{\mu t} x) = A_1 (\log x)^{-\frac{3}{4} + \frac{a}{c^2}} e^{A_2 \sqrt{\log x}} x^{-A_3} \\ (1 + O((\log x)^{-\delta}))$$

as $x \rightarrow \infty$.

Asymptotic behavior of the stock price distribution

The Stein-Stein model.

For $q \geq 0$, $m = 0$, and $\sigma > 0$, there exist $B_1 > 0$, $B_2 > 0$, and $B_3 > 0$ such that for every δ with $0 < \delta < \frac{1}{2}$,

$$D_t(x_0 e^{\mu t} x) = B_1 (\log x)^{-\frac{1}{2}} e^{B_2 \sqrt{\log x}} x^{-B_3} \\ (1 + O((\log x)^{-\delta}))$$

as $x \rightarrow \infty$, and

$$D_t(x_0 e^{\mu t} x) = B_1 \left(\log \frac{1}{x} \right)^{-\frac{1}{2}} e^{B_2 \sqrt{\log \frac{1}{x}}} x^{B_3 - 3} \\ \left(1 + O \left(\left(\log \frac{1}{x} \right)^{-\delta} \right) \right)$$

as $x \rightarrow 0$.

Conclusions. Fat tails.

The distribution density D_t of the stock price in the Hull-White model decays extremely slowly. This is the so-called “fat tail” effect. The function $x \mapsto D_t(x)$ behaves near infinity as $\frac{1}{x^2}$ times logarithmic factors. In addition, the function $x \mapsto D_t(x)$ behaves near zero as $\frac{1}{x}$ multiplied by a logarithmic factor making the density integrable near zero. In a sense, the stock price distribution density in the Hull-White model has the slowest decay possible in comparison with stock price distribution densities in similar stochastic volatility models. It follows from Theorem ?? that the stock price distribution density $D_t(x)$ in the Heston model behaves at infinity roughly as the function x^{-A_3} and at zero as the function x^{A_3-3} . Since

$$C = \frac{\pi^2}{2c^2t} \quad \text{and} \quad A_3 = \frac{3}{2} + \frac{\sqrt{4\pi^2 + c^2t^2}}{2ct}$$

for $b = 0$, we see that in this case the density $D_t(x)$ behaves as the function $x^{-\frac{3}{2} - \frac{\sqrt{4\pi^2 + c^2t^2}}{2ct}}$ near infinity and as the function $x^{-\frac{3}{2} + \frac{\sqrt{4\pi^2 + c^2t^2}}{2ct}}$ near zero.

For the Stein-Stein model, Theorem ?? implies that the function $D_t(x)$ behaves at infinity roughly as the function x^{-B_3} and at zero as the function x^{B_3-3} , where

$$B_3 = \frac{3}{2} + \frac{\sqrt{8C + t}}{2\sqrt{t}} \quad \text{and} \quad C = \frac{1}{2\sigma^2} (tq^2 + t^{-1}r_{tq}^2). \quad (2)$$

Market price of risk and martingale measures

Since the volatility in the Hull-White model is a stochastic process, there is an additional source of uncertainty in the random behavior of the stock price. This uncertainty is controlled by a stochastic process γ_t which is called the market price of the volatility risk.

Assumption: the market price of risk is constant

We assume that $\gamma_t = a$, where a is a number.

Then, under the corresponding martingale measure $\mathbb{P}^{*,a}$, the Hull-White equations can be rewritten in the following form:

$$\begin{cases} dX_t = rX_t dt + Y_t X_t dW_t^* \\ dY_t = (\nu - \xi a) Y_t dt + \xi Y_t dZ_t^*, \end{cases}$$

where W_t^* and Z_t^* are independent standard Brownian motions, and $r > 0$ is the interest rate.

The pricing function

By the martingale property of the discounted stock price with respect to the martingale measure $\mathbb{P}^{*,a}$, the price of a Hull-White European call option at $t = 0$ is given by the following formula:

$$V_0(K) = \mathbb{E}^{*,a} [e^{-rT} (X_T - K)_+].$$

where K is the strike price, and T is the expiration date.

Explicit formula for the pricing function

$$V_0(K) = e^{-rT} \int_K^\infty x D_T(x) dx - e^{-rT} K \int_K^\infty D_T(x) dx.$$

Implied volatility in the Hull-White model

The implied volatility in a general option pricing model is the volatility in the Black-Scholes model for which the

corresponding Black-Scholes price of the option is equal to its price in the model under consideration. Therefore, for $K > 0$ the implied volatility $I(K)$ in the Hull-White model is determined from the equality

$$C_{BS}(K, I(K)) = V_0(K).$$

Implied volatility smile is a mathematical reality

Consider a stochastic volatility model such that the volatility process is independent of the standard Brownian motion driving the stock price equation. Then the volatility function

$$K \mapsto I(K)$$

is convex in a neighborhood of its global minimum

$$K_{min} = x_0 e^{rT}.$$

This is a celebrated result of Renault and Touzi.

Asymptotic behavior of the implied volatility

Large values of the strike price.

$$I(K) = \frac{\sqrt{2 \log K}}{\sqrt{T}} - \frac{\log \log K + \log \log \log K}{2\xi T} + O(1)$$

as $K \rightarrow \infty$.

Small values of the strike price.

$$I(K) = \frac{\sqrt{2 \log \frac{1}{K}}}{\sqrt{T}} - \frac{\log \log \frac{1}{K} + \log \log \log \frac{1}{K}}{2T\xi} + O(1)$$

as $K \rightarrow 0$.

Asymptotic behavior of $I'(K)$ and $I''(K)$

Large values of the strike price.

$$I'(K) = \frac{1}{\sqrt{2T} K \sqrt{\log K}} + O\left(\frac{\log \log K}{K \log K}\right).$$

$$I''(K) = -\frac{1}{\sqrt{2T} K^2 \sqrt{\log K}} + O\left(\frac{\log \log K}{K^2 \log K}\right).$$

as $K \rightarrow \infty$.

Small values of the strike price.

$$I'(K) = -\frac{1}{\sqrt{2T}K\sqrt{\log \frac{1}{K}}} + O\left(\frac{\log \log \frac{1}{K}}{K \log \frac{1}{K}}\right).$$

$$I''(K) = \frac{1}{\sqrt{2T}K\sqrt{\log \frac{1}{K}}} + O\left(\frac{\log \log \frac{1}{K}}{K^2 \log \frac{1}{K}}\right)$$

as $K \rightarrow 0$.

Structural properties of the implied volatility

The implied volatility as a function of the strike price is concave near infinity and convex near zero. These properties combined with the volatility smile effect show that the global shape of the implied volatility function $K \mapsto I(K)$ resembles an infinite dipper.