

Replicating Defaultable Bonds in Black Scholes with Jump To Default

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In this document, we show how to replicate defaultable bonds in the Black Scholes model extended by a jump-to-default.

I Introduction

In Merton[2], the Black Scholes model is extended by allowing the stock price to jump to zero. Under the risk-neutral measure engendered by using the riskfree bond as numeraire, the random default time is exponentially distributed and independent of the standard Brownian motion driving the stock price prior to default. We define a defaultable bond of maturity T as a claim which pays one dollar at T if the stock does not default before T and which pays zero otherwise. In this document, we show how to replicate the payoff to a defaultable bond in this model. The replicating strategy involves dynamic trading in just the stock and a co-terminal call written on the stock. Surprisingly, spanning occurs with just these two assets despite the fact that both Brownian and Poisson drivers are present.

II Assumptions

Consider a fixed time interval $[0, T]$ and assume that the riskfree rate is constant at $r \in \mathbb{R}$ over this period. Let:

$$B_t \equiv e^{-r(T-t)} \tag{1}$$

denote the price at time $t \in [0, T]$ of a bond paying one dollar at T with certainty.

Fix a probability space and let \mathbb{P} denote statistical probability measure. Let S_t denote the spot price of a stock which pays no dividends over $[0, T]$. We assume that S_0 is a known positive constant. Under \mathbb{P} , let S solve the stochastic differential equation (SDE):

$$dS_t = \mu_t S_{t-} dt + \sigma S_{t-} dW_t - S_{t-} dN_t, \quad t \in [0, T], \tag{2}$$

where $\sigma > 0$ is a positive constant. We restrict the process μ so that S can neither explode nor hit zero by diffusion. The Poisson process has statistical arrival rate $\alpha_t \geq 0$. We furthermore require existence and uniqueness of the solution of the SDE. All of these requirements are met if μ and α are constant.

III Black Scholes Model

First consider the famous case where the process α is identically zero. If μ is constant, then the stock price follows geometric Brownian motion. Consider a European call option paying $(S_T - K)^+$ at T with certainty, where $K \geq 0$ is the strike price. It is very well known that for bounded stock price drift μ , the arbitrage-free price of this call at time $t \in [0, T)$ is given by:

$$C_t = N(d_1(S_t, B_t))S_t - KN(d_2(S_t, B_t))B_t, \quad (3)$$

where the functions d_1 and d_2 are defined by:

$$\begin{aligned} d_1(S, B) &\equiv \frac{\ln(\frac{S}{BK}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\ d_2(S, B) &\equiv \frac{\ln(\frac{S}{BK}) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

Under \mathbb{P} , the resulting call dynamics satisfy:

$$dC_t = \mu_t^c C_t dt + N(d_1(S_t, B_t))\sigma S_t dW_t, \quad t \in [0, T], \quad (4)$$

where μ^c causes C to hit its intrinsic value at T and be above it beforehand.

It is also well known that the payoff from a static position in one call option can be replicated by:

1. Holding $N(d_1(S_t, B_t))$ shares at each time $t \in [0, T]$
2. Shorting $KN(d_2(S_t, B_t))$ bonds at each time $t \in [0, T]$.

These positions are simply read off from the Black Scholes call formula (3). Hence, this famous formula is more than a pricing scheme. It is also a replication recipe. Since the position in each of the two assets depends on both asset prices and time, the trading strategy is *dynamic*. It is also well known that this dynamic trading strategy is *self-financing*. In words, the money needed to alter the stock position (which can be negative) is provided by the required change in the bond position.

Finally it is well known that a static position in one call option combined with a short position in $N(d_1(S_t, B_t))$ shares is locally riskless and therefore must earn the riskfree rate r (under \mathbb{P} !). If we scale the call and share positions by the same factor, then the same statement holds, even if the scale factor varies stochastically over time. Suppose that we “solve” (3) for the bond price B :

$$B_t = \frac{C_t}{KN(d_2(S_t, B_t))} - \frac{N(d_1(S_t, B_t))}{KN(d_2(S_t, B_t))} S_t, \quad \text{tin}[0, T]. \quad (5)$$

This formula tells us that the payoff from a static position in one bond can be replicated by:

1. Holding $\frac{1}{KN(d_2(S_t, B_t))}$ calls at each time $t \in [0, T)$
2. Shorting $\frac{N(d_1(S_t, B_t))}{KN(d_2(S_t, B_t))}$ shares at each time $t \in [0, T)$.

Since the positions in call and stock depend on time and the stock and bond prices, the trading strategy is dynamic. This strategy is also self-financing. If we didn’t know the bond price ex-ante, it wouldn’t matter, since if the model holds, we could numerically imply it out from S and C .

In many real world problems, the prices of all three assets are readily observed (say on Bloomberg) and the main objectives are either to hedge or more generally, to make model-based predictions on what one price will be for a given move in the other two. With all of this in mind, let’s see how these results change if we add a jump to default.

IV Black Scholes Model with Jump to Default

We now suppose that the real-world default arrival rate process α is strictly positive at all times. Suppose that there exists a *defaultable bond*, which by definition pays one dollar at T only if no default occurs prior to T . If a default occurs prior to T , then the defaultable bond pays zero. Suppose for now that we can directly observe the market price of the defaultable bond of maturity T (eg. on Bloomberg). For $t < 0$, suppose that this price has enjoyed a constant exponential growth rate $r + \lambda$, where λ is a positive

constant. Based on these observations, we boldly predict that under \mathbb{P} :

$$D_t \equiv e^{-(r+\lambda)(T-t)} 1_{(\tau_1 > t)}, \quad t \in [0, T], \quad (6)$$

where τ_1 is the random jump time of the stock price to zero. Since the only jump allowed in the stock price is the one to zero, τ_1 is the first and only jump time of both the stock price process and the defaultable bond price process. Both processes jump to zero at time τ_1 , which can occur before, at, or after T . Let $\tau \equiv \tau_1 \wedge T$ be the earlier of the default time and maturity.

As we are assuming no arbitrage, we know that there exists a risk-neutral measure \mathbb{Q} engendered by using the riskfree bond as numeraire. It is fashionable nowadays to model credit derivatives by specifying the evolution of various stochastic processes under \mathbb{Q} . Following this trend, suppose that we specify the risk-neutral process for the default arrival rate α as constant at λ over $[0, T]$. As a consequence, the defaultable bond price D would have the \mathbb{P} dynamics given in (6).

Merton[2] showed how to value a call when the Black Scholes model is extended by allowing the stock price to jump to zero at a random time, which has an independent exponential distribution under \mathbb{Q} . The revised pricing formula is:

$$C_t = N(d_1(S_t, D_t))S_t - KN(d_2(S_t, D_t))D_t, \quad t \in [0, T], \quad (7)$$

where the functions d_1 and d_2 are the same, repeated here as:

$$\begin{aligned} d_1(S, D) &\equiv \frac{\ln(\frac{S}{DK}) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\ d_2(S, D) &\equiv \frac{\ln(\frac{S}{DK}) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

A subtle mathematical point is that we now need to know the call value at the point $(S, D) = (0, 0)$ and the functions d_1 and d_2 are not defined at $S = 0$ or $D = 0$. Hence, suppose that we use (7) only for $t \in [0, \tau_1 \wedge T)$ and that we directly define $C_t \equiv 0$ for times t in the possibly empty set $[\tau_1, T)$. Then the call price jumps to zero at the same time that the prices of the defaultable stock and the defaultable bond both vanish. Prior to the earlier of the default time and expiry, all 3 asset prices are positive. As in the

Black Scholes model, the call price can also diffuse into zero at expiry, while in the absence of a jump, the prices of the defaultable stock and bond are strictly positive on $[0, T]$.

The explicit version of Merton's call pricing formula is:

$$C_t = \begin{cases} N(d_1(S_t, D_t))S_t - KN(d_2(S_t, D_t))D_t, & \text{if } t \in [0, \tau) \\ 0 & \text{if } t \in [\tau_1, T]. \end{cases} \quad (8)$$

When considered as a replication recipe, (8) clearly states that the payoff from a static position in one call option can be replicated by:

1. Holding $N(d_1(S_t, D_t))$ (defaultable) shares at each time $t \in [0, \tau]$ and holding zero shares for $t \in (\tau, T)$.
2. Shorting $KN(d_2(S_t, D_t))$ defaultable bonds at each time $t \in [0, \tau]$ and holding zero bonds for $t \in (\tau, T)$.

If $\tau_1 < T$, then for $t \in (\tau_1, T)$, the position in stocks and bonds does not matter since both are worthless. We have specified a holding of zero stocks and zero bonds for concreteness.

The appendix formally proves that the above dynamic trading strategy is both self-financing and replicating. The intuition is clear. If no default occurs over $[0, T]$, then the Black Scholes model is reigning with the riskfree rate given by $r + \lambda$. Hence, we can use standard results to show that the strategy is self-financing and replicating in this case. If a default does occur before T , then the defaultable stock-bond position jumps to zero at the default time, thereby replicating the call value.

It is not surprising that the call payoff can be spanned in the presence of default. Standard arbitrage pricing theory implies that when the Black Scholes model is extended to include a jump of known size (such as the one to default), one can span the payoff of any given equity derivative. What is surprising is that the call payoff can be perfectly replicated by dynamic trading in just two assets when both diffusion and jump components are present. The standard theory requires dynamic trading in three assets eg. the stock, the defaultable bond, and a riskfree bond. Indeed, one cannot replicate the payoff to a European

put option using just dynamic trading in defaultable stocks and bonds. What is special about calls is that prior to expiry, the call value vanishes just when the stock price and bond price both vanish. In contrast, the put takes on a positive value when the stock and bond price vanish. Other equity derivatives sharing the call's vanishing property are power claims paying S_T^p for $p \geq 0$. In fact, we know from the theory of real Laplace transforms (or real Mellin transforms) that these payoffs form a basis for functions $\{f(S); S \geq 0\}$ with $f(0) = 0$. The call payoff $(S - K)^+$ lies in the span of these basis functions. For all such claims, if one uses the stock position to neutralize the Brownian exposure of the portfolio, and if one uses the defaultable bond to finance the required changes in this stock position, then essentially by luck the strategy also neutralizes the Poisson component.

Suppose that for $t \in [0, \tau)$, we “solve” (7) for the defaultable bond price D :

$$D_t = \frac{C_t}{KN(d_2(S_t, D_t))} - \frac{N(d_1(S_t, D_t))}{KN(d_2(S_t, D_t))} S_t \quad t \in [0, \tau). \quad (9)$$

If $\tau_1 < T$, then for $t \in (\tau_1, T)$, the coefficients of C and S in (9) are not defined. Nonetheless, suppose we guess that this formula tells us that the payoff from a static position in one defaultable bond can be replicated by:

1. Holding $\frac{1}{KN(d_2(S_t, D_t))}$ calls at each time $t \in [0, \tau]$ and holding zero calls for $t \in (\tau, T)$.
2. Shorting $\frac{N(d_1(S_t, D_t))}{KN(d_2(S_t, D_t))}$ shares at each time $t \in [0, \tau]$ and holding zero shares for $t \in (\tau, T)$.

This trading strategy in calls and stocks is self-financing, non-anticipating, and does replicate the payoff to the defaultable bond. Again, if we didn't know the defaultable bond price ex-ante, it wouldn't matter, since if the model holds, we could numerically imply it out from S and C .

V Summary and Future Research

We showed that when the Black Scholes model is extended by allowing the stock price to jump to zero at a random time, then the payoff to a static position in one defaultable bond can be replicated by a dynamic

trading strategy in calls and stocks.

Future research can be directed towards generalizing this result by allowing more complicated dynamics for the asset prices. By exploiting both the linear homogeneity of the call payoff and the proportional dynamics of the defaultable stock and bond prices, we expect that the same result will hold if r and λ are each allowed to follow Gaussian processes, which are allowed to be correlated to each other and the stock price. It is not clear whether the parsimony of the hedge survives a restriction to non-negative and non-deterministic processes for r and λ . One would anticipate that the same qualitative results hold under deterministic r and λ , even if the stock volatility and the risk-neutral arrival rate both depend on the paths of traded asset prices. If one adds additional sources of uncertainty, eg. an independent source of variation in volatility and/or the risk-neutral arrival rate, then additional assets will be needed to hedge. It would be interesting to consider newer derivatives such as variance swaps and equity default swaps as hedging instruments.

In general, one can also attempt to consider these results in the context of multiple underlying stocks. One can then attempt a unified framework for CDO's, single name CDS's, index CDS, single name options, index options, single name variance swaps, and index variance swaps. In the interests of brevity, these extensions are best left for future research.

VI Appendix

In this appendix, we will show that the trading strategy in defaultable stocks and bonds described below (8) is self-financing and call replicating. To make these results as applicable as possible, we first focus on the self-financing condition and then later impose the call replicating condition. This whole appendix is just a simple adaptation to the default setting of the insights in Bergman[1].

Let N_t^d and N_t^s respectively denote the number of defaultable bonds and defaultable shares held at

time $t \in [0, T]$. Define V_t as the value of the defaultable bond-stock portfolio at time $t \in [0, T]$:

$$V_t \equiv N_t^d D_t + N_t^s S_t, \quad t \in [0, T]. \quad (10)$$

We say that a strategy $\{(N_t^d, N_t^s), t \in [0, T]\}$ is *self-financing* if:

$$dV_t = N_t^d dD_t + N_t^s dS_t, \quad t \in [0, T]. \quad (11)$$

In words, all changes in portfolio value are sourced from capital gains. No changes in portfolio values are sourced from injections or withdrawals of funds into either the stock or defaultable bond position.

The price processes D and S are both semi-martingales, so applying integration by parts (see eg. Protter[3], pg. 60) to (11):

$$dV_t = N_t^d dD_t + N_t^s dS_t + dX_t, \quad t \in [0, T], \quad (12)$$

where the increments in the external financing process X are given by:

$$dX_t = D_t dN_t^d + d[D, N^d]_t + S_t dN_t^s + d[S, N^s]_t, \quad t \in [0, T]. \quad (13)$$

Comparing (12) to the self-financing condition (11) implies that the latter is equivalent to:

$$D_t dN_t^d + d[D, N^d]_t + S_t dN_t^s + d[S, N^s]_t = 0, \quad t \in [0, T]. \quad (14)$$

We first consider the set of paths for which $\tau_1 > T$. Then the strategy described below (8) becomes:

1. Holding $N(d_1(S_t, D_t))$ (defaultable) shares at each time $t \in [0, T]$
2. Shorting $KN(d_2(S_t, D_t))$ defaultable bonds at each time $t \in [0, T]$.

Instead of showing that this strategy satisfies () and hence is self-financing, we will show that this strategy is implied by (14) and the call replicating condition.

We first note that for the set of paths for which $\tau_1 > T$, D is a continuous process whose sample paths have bounded variation. Hence we have:

$$d[D, N^d]_t = 0, \quad t \in [0, T]. \quad (15)$$

Substituting (15) in (14) implies that the self-financing condition has now simplified to:

$$D_t dN_t^d + S_t dN_t^s + d[S, N^s]_t = 0, \quad t \in [0, T]. \quad (16)$$

To find a solution to this expression, we now restrict ourselves to trading strategies that just depend on the stock price and time. Let $N^d(S, t)$ and $N^s(S, t)$ both be $C^{2,1}$ functions mapping $(0, \infty) \times [0, T]$ into \mathbb{R} . Then:

$$N_t^d = N^d(S_t, t) \quad N_t^s = N^s(S_t, t), \quad t \in [0, T]. \quad (17)$$

Rather than work directly with $N^d(S, t)$, we may define a $C^{2,1}$ function by:

$$V(S, t) \equiv N^d(S, t)e^{-(r+\lambda)(T-t)} + N^s(S, t)S, \quad S > 0, t \in [0, T]. \quad (18)$$

Equations (6), (10), (17), and (18) imply that:

$$V_t = V(S_t, t), \quad t \in [0, T]. \quad (19)$$

For the set of paths for which $\tau_1 > T$, the trading strategy $\{N_t^s, N_t^d, t \in [0, T]\}$ is determined by a specification of the functions $N^s(S, t)$ and $V(S, t)$ on the common domain $S > 0, t \in [0, T]$. We now seek to determine what restrictions the self-financing condition (16) places on this pair of functions. Equations (17) and (18) imply that:

$$N_t^d = [V(S_t, t) - N^s(S_t, t)S_t]e^{(r+\lambda)(T-t)}, \quad t \in [0, T]. \quad (20)$$

Substituting (20) in (16) implies:

$$D_t d\{[V(S_t, t) - N^s(S_t, t)S_t]e^{(r+\lambda)(T-t)}\} + S_t dN_t^s + d[S, N^s]_t = 0, \quad t \in [0, T]. \quad (21)$$

From the product rule and (1), this simplifies to:

$$-(r + \lambda)[V(S_t, t) - N^s(S_t, t)S_t]dt + d\{V(S_t, t) - N^s(S_t, t)S_t\} + S_t dN_t^s + d[S, N^s]_t = 0, \quad t \in [0, T]. \quad (22)$$

Applying integration by parts:

$$-(r + \lambda)[V(S_t, t) - N^s(S_t, t)S_t]dt + dV(S_t, t) - N^s(S_t, t)dS_t - S_t dN_t^s - d[S, N^s]_t + S_t dN_t^s + d[S, N^s]_t = 0, t \in [0, T]. \quad (23)$$

Cancelling the last four terms on the LHS and using Itô's formula implies that for $t \in [0, T]$:

$$\left\{ \frac{\sigma^2 S_t^2}{2} \frac{\partial^2}{\partial S^2} V(S_t, t) + \frac{\partial}{\partial t} V(S_t, t) - (r + \lambda)[V(S_t, t) - N^s(S_t, t)S_t] \right\} dt + \left[\frac{\partial}{\partial S} V(S_t, t) - N^s(S_t, t) \right] dS_t = 0. \quad (24)$$

Equation (24) can hold only if the coefficient of dS_t is zero, which implies that delta-hedging is necessary:

$$N^s(S_t, t) = \frac{\partial}{\partial S} V(S_t, t), \quad t \in [0, T]. \quad (25)$$

Furthermore, equation (24) can hold only if the coefficient of dt is also zero. Substituting in (25) implies:

$$\frac{\sigma^2 S_t^2}{2} \frac{\partial^2}{\partial S^2} V(S_t, t) + \frac{\partial}{\partial t} V(S_t, t) - (r + \lambda)V(S_t, t) + (r + \lambda)S_t \frac{\partial}{\partial S} V(S_t, t) = 0, \quad t \in [0, T]. \quad (26)$$

Hence, the following PDE must govern the value function $V(S, t)$:

$$\frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} V(S, t) + \frac{\partial}{\partial t} V(S, t) - (r + \lambda)V(S, t) + (r + \lambda)S \frac{\partial}{\partial S} V(S, t) = 0. \quad S > 0, t \in [0, T]. \quad (27)$$

Suppose that we further impose the call payoff replication condition:

$$V(S, T) = (S - K)^+, \quad S \geq 0, \quad (28)$$

along with the boundary conditions:

$$\lim_{S \uparrow \infty} V_s(S, t) = 1, \quad t \in [0, T] \quad (29)$$

$$\lim_{S \downarrow 0} V_s(S, t) = 0, \quad t \in [0, T]. \quad (30)$$

Then there exists a unique solution which is given by:

$$V(S, t) = N(d_1(S, D(t)))S - KN(d_2(S, D(t)))D(t), \quad S > 0, t \in [0, T], \quad (31)$$

where:

$$D(t) \equiv e^{-(r+\lambda)(T-t)} \quad t \in [0, T], \quad (32)$$

and recall that the functions d_1 and d_2 are defined by:

$$d_1(S, D) \equiv \frac{\ln\left(\frac{S}{DK}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2(S, D) \equiv \frac{\ln\left(\frac{S}{DK}\right) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}.$$

Hence, from (25), the share position is given by:

$$N^s(S, t) = \frac{\partial}{\partial S} V(S, t) = N(d_1(S, D(t))), \quad S > 0, t \in [0, T]. \quad (33)$$

From (20), the defaultable bond position is given by:

$$N^d(S, t) = [V(S, t) - N^s(S, t)S]e^{(r+\lambda)(T-t)} = -KN(d_2(S, D(t))), \quad S > 0, t \in [0, T]. \quad (34)$$

We now consider the set of paths for which $\tau_1 \leq T$. Since the time at which default occurs is not known until it happens, the strategy employed is:

1. Holding $N(d_1(S_t, D_t))$ (defaultable) shares at each time $t \in [0, \tau_1]$ and holding zero shares for $t \in (\tau_1, T)$.
2. Shorting $KN(d_2(S_t, D_t))$ defaultable bonds at each time $t \in [0, \tau_1]$ and holding zero bonds for $t \in (\tau_1, T)$.

We have already shown that this strategy is self-financing for $t \in [0, \tau_1)$. For $t \in [\tau_1, T)$, we have:

$$D_t dN_t^d + d[D, N^d]_t + S_t dN_t^s + d[S, N^s]_t = 0, \quad t \in [0, T], \quad (35)$$

simply because $S = D = 0$. In fact, any strategy employed on the time interval $t \in [\tau_1, T)$ is self-financing. It is also true that any strategy employed over this time interval results in a terminal value of $V_T = 0$ when $\tau_1 \leq T$. Fortunately, this is the call payoff when $\tau_1 \leq T$ since $S_T = 0$. We have therefore shown that the trading strategy described below (8) is both self-financing and call replicating even if $\tau_1 \leq T$. As we have already shown that this strategy is self-financing and call replicating for paths where $\tau_1 > T$, we may conclude that the strategy is self-financing and call replicating for all paths.

References

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