

# The Paradox of Asset Pricing

Introductory Remarks

On the **predictive power** of modern finance:

*“It is a very beautiful line of reasoning.  
The only problem is that perhaps it is not true.  
(After all, nature does not have to go along  
with our reasoning.)”*

(Richard P. Feynman, in *Lectures on Physics*.)

(E.g., CAPM and other linear asset pricing models: widely used *as if* they're right!?)

The position we're going to take:

*Theory is not necessarily to blame  
for the poor scientific record of asset pricing,  
but empirical methodology*

# The Paradox of Asset Pricing

## I. The Principles of Asset Pricing...

*... needed to understand the work of empiricists.*

*So, we'll ignore, among other things, heterogeneous information!*

## 6. Consumption-Based Asset Pricing Models

The first-order conditions of agents' intertemporal investment-consumption are referred to as *stochastic Euler equations*. **These form an asset pricing model: in equilibrium, they ought to be satisfied for all agents.**

Problem: we don't have [historical, field] data at the individual level.

Let's be bold [Lucas] and **assume all investors are alike**. Then aggregate per-capita and private consumption coincide, **so the stochastic Euler equations are true for aggregate consumption as well:**

$$\delta E \left[ \frac{\frac{\partial \tilde{u}(c'_A)}{\partial c'}}{\frac{\partial \tilde{u}(c_A)}{\partial c}} R_n | x \right] = 1. \quad (1)$$

## A Special Case: The Rubinstein dynamic CAPM Model

- Assume that the representative agent has log utility.
- The following is easy to check:

$$c_A = (1 - \delta)x_A; \quad c'_A = (1 - \delta)x'_A.$$

- In equilibrium,

$$\frac{x'_A}{\delta x_A} = R_M.$$

So:

$$E\left[\frac{\delta x_A}{x'_A} R_n | x\right] = E\left[\frac{1}{R_M} R_n | x\right] = 1. \quad (2)$$

... independent of return stochastics!

Joint lognormality of individual and market returns implies:

$$\mu_{n,x} - \ln R_F = 2\tilde{\beta}_{x,n}^M (\mu_{M,x} - \ln R_F) - \frac{1}{2}\sigma_{n,x}^2. \quad (3)$$

## 7. Asset Pricing Theory: The Bottom Line

$$E[AR_n|x] = 1, \quad (4)$$

where  $A$  measures aggregate risk.

E.g., Rubinstein:  $A = \frac{1}{R_M}$ .

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## II. Empirical Methodology

Point of departure:

- Pick a measure of aggregate risk  $A$ . Then this is what we need to test:

$$\boxed{E^m[AR_n|x] = 1.} \quad (5)$$

- To test this, we need to estimate beliefs used to form expectations (“ $m$ ”).
- The theory puts little restrictions on beliefs, *except Lucas-RE* – see below.
- So, to test the theory, we should really be asking agents what they expected ex ante.
- That’s not what’s being done...

What's being done... is to link beliefs to actual outcomes:

*Beliefs are "correct" (The Efficient Markets Hypothesis);  
Equivalently, beliefs are confirmed by the actual outcomes.*

Which really means:

1. Beliefs are **unbiased** expectations of the true probabilities;
2. The true probabilities can be estimated by simple **averaging** of outcomes.

Here is what these things mean mathematically/statistically:

1. We can drop the  $m$ :

$$E[AR_n|x] = 1.$$

2. We need a law of large numbers, and for that, we need *stationarity*, so that, for  $A_tR_{n,t}$ ,  $t = 1, 2, \dots$ :

$$\frac{1}{T} \sum_{t=1}^T A_tR_{n,t} \rightarrow E[AR_n](= 1).$$

Ignoring the conditioning on the state variable  $x$ , these assumptions allow for a **simple method of moments test**:

Are

$$\frac{1}{T} \sum_{t=1}^T A_t R_{n,t}$$

equal to

1

?

## Remarks:

- A  $t$ -test would also require independence and normality; an (asymptotic)  $z$ -test would require the right “mixing” – decay in dependence, but let’s not worry about that...
- All tests are generalizations of this simple method of moments test.

One can do a *reductio ad absurdum*:

The approach implies that...

- ... the high average returns on Apple stock over the last five years (37.4% p.a.) relative to IBM (-2.3% p.a. ) reflect the higher risk of holding Apple stock,
- and the negative average returns on the Nikkei index since 1990 against the high positive average return on the SP 500 index reflect the much lower risk of the former.

Aside from the issues of correct beliefs and stationarity, the empirical methodology is **remarkably parsimonious and robust**: it is possible to test asset pricing theory with **minimal knowledge about**:

- the data generating process;
- information that agents may have had.

(...although these features are not always exploited!)

**Any new test that relaxes some of the assumptions should aim at the same features!**

# The Paradox of Asset Pricing

## V. Improving Empirical Methodology

*... where we develop a new methodology for testing asset pricing models that allows the market to hold mistaken expectations at times, but we'll still require it to learn as under EMH. We will call it the Hypothesis Of Efficiently Learning Markets (ELM).*

## Preliminary Remarks:

- Traditional empirical methodology requires a tight link between ex-ante beliefs in the marketplace and what factually is in the dataset of the empiricist ex-post.
- There are a variety of reasons why this link is too tight.
  1. The market may have had biased beliefs because, e.g., lack of experience (e.g., IPOs in start-up airline companies during deregulation).
  2. The empiricist inadvertently collected a biased sample, where an expected event happened not to have occurred (a *Peso problem*; e.g., the devaluation of the currency did not happen in the time span under investigation).
- Can we do something about this *without losing the parsimony* of the traditional approach?

## Overview

1. A (Surprising) Property of Bayesian Learning
2. Digital Option Prices Under ELM and Risk Neutrality
3. Limited-Liability Securities Prices Under ELM and Risk Neutrality
4. Re-Introducing Risk Aversion and You Favorite Asset Pricing Model

# 1. Bayesian Learning Re-Visited

Use the standard set-up:

- Let  $t$  index time.  $t = 0, 1, \dots, T$ .
- Unknown real-valued parameter  $V$ , to be revealed at  $T$ .
- (Arbitrary) prior belief  $\lambda_0(\cdot)$ .
- Signals  $x_t$  at  $t = 1, \dots, T - 1$  with known likelihood:

$$l_t(x_t|x_{t-1}, V).$$

Learning using the rules of conditional probability (Bayes' law):

$$\lambda_t(V) = \frac{l_t(x_t|x_{t-1}, V)\lambda_{t-1}(V)}{\int l_t(x_t|x_{t-1}, v)\lambda_{t-1}(v)dv}, \quad (t = 1, 2, \dots, T - 1). \quad (6)$$

Beliefs form a martingale, hence learning is "rational" (Doob):

$$E^m[\lambda_t(V)|x_{t-1}, \dots, x_1] = \lambda_{t-1}(V). \quad (7)$$

Note:

$$E^m[\lambda_t(V)|x_{t-1}, \dots, x_1] = \int \int \lambda_t(V) l_t(x|x_{t-1}, v) \lambda_{t-1}(v) dx dv.$$

Now consider

$$E[\cdot | x_{t-1}, V] (= E[\cdot | x_{t-1}, \dots, x_1, V]).$$

(The Markov assumption justifies the equality in parentheses.)

In particular, study

$$E\left[\frac{\lambda_{t-1}(V^*)}{\lambda_t(V^*)} | x_{t-1}, V\right].$$

Technicality: we have to ensure that this conditional expectation exists, i.e., that

$$E\left[\left|\frac{\lambda_{t-1}(V^*)}{\lambda_t(V^*)}\right|\right] < \infty.$$

For simplicity, assume that there exists  $\epsilon > 0$  such that, for all  $V^*$ , and for  $t = 0, 1, \dots, T - 1$ ,

$$P\{\lambda_t(V^*) \geq \epsilon | V\} = 1. \tag{8}$$

*Call this the “No Early Exclusion Hypothesis” (NEEH)*

... which is violated, for instance, if information flows can be represented in terms of a recombining binomial tree!

Now, under NEEH:

$$E\left[\frac{\lambda_{t-1}(V^*)}{\lambda_t(V^*)} \mid x_{t-1}, V^*\right] = 1. \quad (9)$$

Proof:

$$\begin{aligned} & E\left[\frac{\lambda_{t-1}(V^*)}{\lambda_t(V^*)} \mid x_t, V^*\right] \\ &= \int \frac{\lambda_{t-1}(V^*)}{\lambda_t(V^*)} l_t(x \mid x_{t-1}, V^*) dx \\ &= \int \frac{\lambda_{t-1}(V^*) \int l_t(x \mid x_{t-1}, v) \lambda_{t-1}(v) dv}{l_t(x \mid x_{t-1}, V^*) \lambda_{t-1}(V^*)} l_t(x \mid x_{t-1}, V^*) dx \\ &= \int \int l_t(x \mid x_{t-1}, v) dx \lambda_{t-1}(v) dv \\ &= 1. \end{aligned}$$

## 2. Application To Digital Option Prices (Under Risk Neutrality)

- Digital option pays \$1 if  $V = V^*$ , 0 otherwise.
- So,

$$P_t = E^m[1_{\{V=V^*\}} | x_t, x_{t-1}, \dots] = \lambda_t(V^*).$$

The following follows immediately from (9):

$$E\left[\frac{P_{t-1}}{P_t} \mid x_{t-1}, V = V^*\right] = 1. \quad (10)$$

*As in traditional tests, readily testable...*

1. No additional parameters,
2. One can ignore information: for a smaller conditioning vector  $x_{b,t-1}$ :

$$E\left[E\left[\frac{P_{t-1}}{P_t} \mid x_{t-1}, V = V^*\right] \mid x_{b,t-1}, V = V^*\right] = E\left[\frac{P_{t-1}}{P_t} \mid x_{b,t-1}, V = V^*\right] = 1.$$

### 3. Application To Limited-Liability Securities Prices (Under Risk Neutrality)

- Payoff: either (i) *no default/in-the-money*, in which case payoff =  $V > 0$ ; or (ii) if  $V \leq 0$ , *default/out-of-the-money*, in which case payoff = 0.
- $P_t = E^m[V 1_{\{V > 0\}} | x_t, x_{t-1}, \dots]$ .
- Assumed *unbiased conditional expectations* (UCE):

$$E^m[V | x_t, x_{t-1}, \dots, V > 0] = E[V | x_t, x_{t-1}, \dots, V > 0].$$

Under NEEH and UCE:

$$E\left[\frac{P_t - P_{t-1}}{P_t} V \mid x_{t-1}, V > 0\right] = 0. \quad (11)$$

*Again, this is readily verifiable... on substantially biased samples!*

*And:*

1. No additional parameters relative to traditional tests,
2. One can again ignore information, as in traditional tests.

Proof:

$$\begin{aligned} & E\left[\frac{P_t - P_{t-1}}{P_t} V | x_{t-1}, V > 0\right] \\ &= E[V | x_{t-1}, V > 0] - E\left[\frac{P_{t-1}}{P_t} E[V | x_t, x_{t-1}, \dots, V > 0] | x_{t-1}, x_{t-2}, \dots, V > 0\right] \\ &= E[V | x_{t-1}, V > 0] - E\left[\frac{\lambda_{t-1}(\{V > 0\})}{\lambda_t(\{V > 0\})} \frac{E[V | x_{t-1}, \dots, V > 0]}{E[V | x_t, x_{t-1}, \dots, V > 0]} \right. \\ &\quad \left. E[V | x_t, x_{t-1}, \dots, V > 0] | x_{t-1}, x_{t-2}, \dots, V > 0\right] \\ &= E[V | x_{t-1}, V > 0] - E\left[\frac{\lambda_{t-1}(\{V > 0\})}{\lambda_t(\{V > 0\})} E[V | x_{t-1}, \dots, V > 0] | x_{t-1}, x_{t-2}, \dots, V > 0\right] \\ &= E[V | x_{t-1}, x_{t-2}, \dots, V > 0] \left(1 - E\left[\frac{\lambda_{t-1}(\{V > 0\})}{\lambda_t(\{V > 0\})} | x_{t-1}, x_{t-2}, \dots, V > 0\right]\right) \\ &= 0. \end{aligned}$$

## 4. Re-Introducing Risk Aversion And Your Favorite Asset Pricing Model

... a *no-brainer* for somebody familiar with mathematical finance:

- Start from your asset pricing model:  $E^m[A_t R_{n,t} | x_{t-1}] = 1$ . (Note superscript!)
- Re-write this in terms of prices:

$$E^m[A_t P_t | x_{t-1}] = P_{t-1}.$$

- Adding past information does not hurt (Markov):

$$E^m[A_t P_t | x_{t-1}, x_{t-2}, \dots] = P_{t-1}.$$

(C'd)

- Apply this to the pricing at times  $T$  and  $T - 1$ :

$$E^m[A_T P_T | x_{T-1}, x_{T-2}, \dots] = P_{T-1}.$$

- Since  $A_{T-1}, A_{T-2}, \dots, A_0$  are all in the market's information set at time  $T - 1$ , we could as well have written:

$$E^m[A_T A_{T-1} A_{T-2} \dots A_0 P_T | x_{T-1}, x_{T-2}, \dots] = A_{T-1} A_{T-2} \dots A_0 P_{T-1}.$$

- An analogous operation can be done at any prior point in time:

$$E^m[A_t A_{t-1} \dots A_0 P_t | x_{t-1}, x_{t-2}, \dots] = A_{t-1} \dots A_0 P_{t-1}.$$

So, define the **deflated price**  $\tilde{P}_t = A_t A_{t-1} \dots A_0 P_t$ , then:

$$E^m[\tilde{P}_t | x_t, x_{t-1}, \dots] = \tilde{P}_{t-1},$$

and, iterating, one obtains:

$$\tilde{P}_t = E^m[\tilde{P}_T | x_t, x_{t-1}, \dots], \tag{12}$$

with

$$\tilde{P}_T = A_T A_{T-1} \dots A_0 P_T = A_T A_{T-1} \dots A_0 V \mathbf{1}_{\{V > 0\}}.$$

Define:

$$\tilde{V} = A_T A_{T-1} \dots A_0 V, \quad (13)$$

then:

$$E\left[\frac{\tilde{P}_t - \tilde{P}_{t-1}}{\tilde{P}_t} \tilde{V} \mid x_{t-1}, x_{t-2}, \dots, \tilde{V} > 0\right] = 0. \quad (14)$$

so that our restrictions obtain for the “tilde” variables!

## **Final remark:**

Unlike some recent work by Brennan, Xhina, Veronesi, Lewellen and others, we're not casting the learning problem in terms of an unknown dividend process, but in terms of the value of the security at some future point of time – ours is more general.

In a nutshell...

- We have derived a set of restrictions that **obtain even if the market has biased priors (but uses the correct likelihood)**.
- These restrictions **can (need to) be tested on biased samples**.
- The test is **parsimonious**, like traditional tests: (i) robust to ignoring information that the market had at the time of decision making, (ii) no more parameters (e.g., prior beliefs) need to be estimated.

# The Paradox of Asset Pricing

## VI. Re-Visiting The Historical Record

1. IPO aftermarket pricing
2. Index call options

## 2. Index Call Options

*(Based on Bossaerts (2004))*

- SPX 1991-95: big price run-up during period of lowest volatility (1995)!?
- Study (real) daily closing prices of SPX option contracts.
- At-the-money 5 weeks before expiration; nearest-maturity; followed for 4 weeks.
- Risk adjustment using (i) Rubinstein's log utility model; (ii) Rubinstein's generalization to power utility:

$$A_t = \left( \frac{1}{R_{M,t}} \right)^\gamma .$$

- This is not derivatives pricing; options are priced directly.
- Relaxation of beliefs relative to traditional tests: markets may have gotten the frequency wrong that at-the-money call options expired in the money.

Using Rubinstein's log utility model:

Selection Bias	Return Measure	<i>N</i>	Daily Average (%)	Intercept	Slope
<u>All Contracts</u>	Standard	1102	-0.925* (-1.767)	-0.031** (-5.526)	-0.954** (-3.455)
<u>Winning Contracts</u>	Standard	760	1.102* (1.608)		
	Modified	760	-2.109** (-3.258)		
	Weighted Standard	760	0.069** (3.506)		
	Weighted Modified	760	-0.002 (-0.094)	-0.000 (-1.573)	-0.027** (-2.844)

*(Projections use option richness [inverted: log (strike/index)] as explanatory variable.)*

Using Rubinstein's power-utility model with  $\gamma = 5$ :

Selection Bias	Return Measure	<i>N</i>	Daily Average (%)	Intercept	Slope
<u>All Contracts</u>	Standard	1102	-1.407** (-2.897)	-0.033** (-6.439)	-0.954** (-3.830)
<u>Winning Contracts</u>	Standard	760	0.380 (0.586)		
	Modified	760	-2.168** (-3.759)		
	Weighted Standard	760	0.028** (2.135)		
	Weighted Modified	760	-0.019 (-1.545)	-0.000** (-2.162)	-0.010 (-1.561)

## Conclusion

1. Our test has power – it rejects where you expect.
2. Risk adjustment can be explained in terms of simple models once obvious biases in beliefs are taken into account.
3. We need more applications.