Convertible Bonds & Swing Contracts as Dynkin Games: A Monte Carlo Approach

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Stanford April 20, 2007
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Why Convertible Bonds

- **Hybrid** derivative:
  - **equity component**: stock shares
  - **fixed income component**: interest coupon payments

- **Upside** in case stock price goes up
- **Little or No Down Side**: bond protection
- **Was** extremely popular and very actively traded
  - Set back in May 2005 (GM/Ford)
- **Volume** recently up again
Typical Corporate Bond Scenario

In general **Seller**
- **Collects nominal** (loan amount) at inception
- **Pays coupons** (interest) at regular time intervals
- **Returns nominal** at maturity

In general **Bond Holder**
- **Pays nominal** (loan amount) upfront
- **Receives coupons** (interest) at regular time intervals
- **Retrieves nominal** at maturity

In case of **Default**
- Bond holder gets **recovery** (proportion of nominal) at time of **default** (before maturity)
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Extra Features of Convertible Bonds

Seller can at a time of his choosing (stopping time)
- Return the nominal (loan amount)
- Stop paying (interest) coupons

Bond Holder can at a time of his choosing (stopping time)
- Request nominal and walk away (game over)
- Convert loan title into company shares

The contract ends the first time one of the two counterparties exercises his/her right
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Convertible Bonds

- Corporate (defaultable) Bond
- Option to exchange for a given number of shares
- Complex Indentures
  - Put / Redemption provision
  - Call provision
  - Put / Redemption protection
  - Call protection
  - Call notice
  - Treatment at Maturity
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Converts
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Typical Evolution

- **Bond holder** chooses a strategy $\tau_b$ (stopping time)
- **Bond seller** chooses a strategy $\tau_s$ (stopping time)

$$ R(\tau_b, \tau_s) = \begin{cases} 
L_{\tau_b}, & \text{whenever } \tau_b \leq \tau_s \text{ or } \tau_b = \tau_s < N \\
\xi, & \text{whenever } \tau_b = \tau_s = N \\
U_{\tau_s}, & \text{whenever } \tau_b > \tau_s 
\end{cases} $$

$R$ present value of payments to bond holder $B$ from seller $S$

- $L$ is $B$ **converts first**
- $U$ if $S$ **calls** the bond **first**
- $\xi$ if neither party exercises the option before maturity
Mathematical Problem

- Bond holder $B$ tries to maximize $\mathbb{E}\{R(\tau_b, \tau_s)\}$
- Issuer $S$ tries to minimize $\mathbb{E}\{R(\tau_b, \tau_s)\}$

What is the value of such a contract?
Current Pricing not Satisfactory

- Traditional **Models** based on *restrictive* assumptions
  - Traditional (split straight bond / option) only an approximation
  - Asset based ("structural approach") require precise asset/liability information

- **Current Implementations** based on **tree** models & **PDE** solvers
  - do not match market prices
  - do not match market deltas
  - .................................. 

Room for experimentation with new ideas & new pricing algorithms
Convert Trader’s Wish List

- One program for **quick pricing** (with prices in line with those from third party providers)
- One **robust model implementation** including more of the bond indentures and with implementations for Risk Management, market makers and possibly proprietary traders (quite likely slower than first program)
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Monte Carlo Approach

Need to generate **Monte Carlo Scenarios**

- Need a model for **Equity Dynamics** (though not a Merton like "Structural Approach")
  - Started with a Geometric Brownian Motion
  - MC allows any SDE for dynamics (including stochastic volatility, local volatility, .... models)
  - Use **exact simulation** or Euler or higher order scheme to generate Monte Carlo scenarios

- Need a model for **Default Intensity** (Cox process in the spirit of "Reduced Form Approach")
  - Need to be able to generate scenarios for the intensity
  - Compute the running integral
  - Draw (independently) an exponential random variable
  - Scenario for the time of default given by first time intensity running integral crosses above the exponential variate.
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Done via the construction of **two random sequences**

- \( \{L_j\}_j \) gives the present value (PV) at time \( t = 0 \) of the **cumulative** cash flows to the holder, before-and-including time \( t = j\Delta t \), should she decide to exercise her right(s) (**conversion, redemption/put, ...**) at time \( j\Delta t \), while issuer has not exercised any of her options yet.

- \( \{U_j\}_j \) gives PV of the **cumulative** cash flows to the holder from the issuer before and including time \( t = j\Delta t \) should the issuer decide to exercise her right(s) to **call** the bond at time \( j\Delta t \) while holder has not exercised any of her options yet.
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Typical MC Scenario for \( \{L_j\}_j \)

- **Before default** (i.e. \( j\Delta t < \tau \))
  - \( L_j = AI + PV \) of coupon payments up to time \( j\Delta t \) if \( j\Delta t < T_{\text{conv}} \)
    - conversion protection threshold
  - \( L_j = AI + PVCP[j] + \text{ConvRat} \times S[j\Delta t] \)
    - if \( j\Delta t > T_{\text{conv}} \) (no redemption possible)
  - \( L_j = AI + PVCP[j] + \max\{\text{ConvRat} \times S[j\Delta t], P\} \)
    - if \( j\Delta t > T_{\text{conv}} \) and put possible

- **After default** (i.e. \( j\Delta t \geq \tau \))
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Typical MC Scenario for $\{U_j\}_j$

- **Before default** (i.e. $j\Delta t < \tau$)
  - $U_j \equiv \infty$ up to the time $T_{\text{call}}$ of hard call protection
  - $U_j = AI + PVCP[j] + \max\{\text{ConvRat} \ast S[j\Delta t], P_{\text{call}}\}$ if $j\Delta t > T_{\text{call}}$ (call possible)
  - Include Make Whole provisions
  - Implement Soft Call Protection
  - Include Call Notice of Redemption Period provisions
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- **After default** (i.e. $j\Delta t \geq \tau$)
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Notice

$$L_j \leq U_j \quad \text{and} \quad L_N = U_N = \xi.$$
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Bond holder chooses a strategy $\tau_b$ (stopping time)
Bond seller chooses a strategy $\tau_s$ (stopping time)

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  U_{\tau_s}, & \text{whenever } \tau_b > \tau_s 
\end{cases}$$

In words: \textit{PV of payments to bond holder B from seller S}

- $L$ is $B$ converts (strictly) before maturity and no later than $S$ calls the bond
- $U$ if $S$ calls the bond first
- $\xi$ if neither party exercises her option before maturity
Dynkin Games of Timing

- Bond holder $B$ tries to maximize $\mathbb{E}\{R(\tau_b, \tau_s)\}$
- Issuer $S$ tries to minimize $\mathbb{E}\{R(\tau_b, \tau_s)\}$

Framed this way,

Convertible Bond Set Up = Dynkin Game (of Timing) Set Up
Game Value Functions

Game upper value

\[ \overline{V} = \sup_{\tau_b} \inf_{\tau_s} \mathbb{E}\{ R(\tau_b, \tau_s) \} \]

Game lower value

\[ \underline{V} = \inf_{\tau_s} \sup_{\tau_b} \mathbb{E}\{ R(\tau_b, \tau_s) \} \]

Interpretation

- For any seller’s call strategy \( \tau_s \), holder chooses \( \tau_b \) which maximizes her expected reward
  \[ \sup_{\tau_b} \mathbb{E}\{ R(\tau_b, \tau_s) \} \]  \((*)\)
  if seller is prudent, she chooses \( \tau_s \) to minimize \((*)\), hence the lower bound \( \underline{V} \).
- This \textit{min-max} strategy guarantees that expected payment to the bond holder is at least \( \underline{V} \).
- Exchange roles of issuer and holder to get a \textit{max-min} strategy for which, whatever convert strategy the holder uses, the expected payment to the holder cannot exceed \( \overline{V} \).
For each \( n \) define random variables \( V_n \) and \( \overline{V}_n \) by

\[
V_n = \inf_{\tau_b \in S_n} \sup_{\tau_s \in S_n} \mathbb{E}_n\{R(\tau_b, \tau_s)\}
\]

and

\[
\overline{V}_n = \sup_{\tau_s \in S_n} \inf_{\tau_b \in S_n} \mathbb{E}_n\{R(\tau_b, \tau_s)\}
\]

- \( S_n \) set of \( \{\mathcal{F}_n\}_n \)-stopping times \( \tau \), \( n \leq \tau \leq N \)
- \( \mathbb{E}_n \) conditional expectation w.r.t. \( \mathcal{F}_n \), i.e. \( \mathbb{E}_n\{ \cdot \} = \mathbb{E}\{ \cdot | \mathcal{F}_n \} \).
Main Results of the Theory

\[ V_n = \overline{V}_n, \quad 0 \leq n \leq N \]

Dynamic programming principle (backward induction)

\[ V_n = \begin{cases} 
L_n, & \text{if } \mathbb{E}_n\{V_{n+1}\} < L_n \\
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\end{cases} \]

starting from the terminal condition \( V_N = \xi \).

Minimal optimal stopping times

\[ \tau_b^* = \inf\{n \geq 0; \ V_n \leq L_n\} \]

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\[ \tau^*_s = \inf\{n \geq 0; \ V_n \geq U_n\}. \]
Conclusion: we have a Pricing Algorithm

Like in the case of American options

- Choose your favorite regression method
- Compute value functions $V_n$ backward-in-time starting from time $n = N$ down to $n = 0$.
- Read off the convertible bond price as the value function at time $n = 0$
- Compute the optimal exercise times scenario-by-scenario in a forward-in-time pass through the scenario
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Choice of a Model with Equity dependent Spreads

Choose **default intensity** function of the **underlying spot**

- \( \lambda_t = \lambda(S_t) \)

- \( \lambda(x) = \beta_0 x^{-\beta_1} \)
  - (Andersen-Buffum, Ayache-Forsyth-Vetzal, Davis-Lischka, Duffie-Singleton, Muromachi, Takahashi-Kobayashi-Nakagawa, Linetsky, · · ·)

- \( \lambda(x) = \beta_0 e^{-\beta_1 x} \)
  - (Bloch-Miralles, Arvanitis-Gregory, · · ·)

**In any case:**

**ONE FACTOR MODEL**
Choice of a Model with Equity dependent Spreads

Choose **default intensity** function of the **underlying spot**

$$\lambda_t = \lambda(S_t)$$

- $$\lambda(x) = \beta_0 x^{-\beta_1}$$
  - (Andersen-Buffum, Ayache-Forsyth-Vetzal, Davis-Lischka, Duffie-Singleton, Muromachi, Takahashi-Kobayashi-Nakagawa, Linetsky, · · · )

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In any case:

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- \( \lambda (x) = \beta_0 e^{-\beta_1 x} \)
  
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**In any case:**

**ONE FACTOR MODEL**
Calibration

Local volatility function $\sigma(t, S_t)$ function of the underlying spot
- grab equity option prices
- construct a local volatility surface
- replace constant $\sigma$ of GBM by $\sigma(t, S_t)$ in Euler or Milstein or ...

**NOT DONE**

Intensity parameters $\beta_0, \beta_1$ to match market CDS spread curve
- Use a Levenberg-Marquardt form of least squares calibration
- Use MC to compute CDS spreads from model

Works OK but not robust
Local volatility function \( \sigma(t, S_t) \) function of the underlying spot

- grab equity option prices
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**Works OK** but not robust
Regression

$\mathcal{F}_n$ contains information about $S_k \textbf{ AND }$ events $\{\tau \leq k\}$ for $k \leq n$

Add a jump to 0 at default: set $S_t = 0$ when $t \geq \tau$
(i.e. add a jump martingale term to SDE for $S_t$)

$$\tilde{S}_n = \begin{cases} S_n, & \text{if } n < \tau \\ 0, & \text{if } n \geq \tau \end{cases}$$

In the backward induction

Regress $V_{n+1}$ against $\tilde{S}_n$ instead of $\mathcal{F}_n$!
\[ \mathcal{F}_n \] contains information about \( S_k \) AND events \( \{ \tau \leq k \} \) for \( k \leq n \)

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In the backward induction

**Regress** \( V_{n+1} \) against \( \tilde{S}_n \) instead of \( \mathcal{F}_n \)!
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In the backward induction

**Regress $V_{n+1}$ against $\tilde{S}_n$ instead of $F_n$ !**
Looking at Data

In the \((S_n, V_{n+1})\) - plane

- blob of points above \(S_n = 0\) (default prior to or at \(n\))
- mild "hockey-stick" shape or linear cloud above \(S_n > 0\)

Easy way-out:

- Plain average for \(S_n = 0\)
- Plain Least Squares **Piecewise Linear Regression** of \(V_{n+1}\) against \(S_n\) when \(S_n > 0\)!
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Looking at Data: Sirius

Sirius, 3.5%, June 1, 2008, J=J_{max}=324
Looking at Data: Sirius

Sirius, 3.5%, June 1, 2008, J=50
Looking at Data: Bearingpoint

Bearingpoint, 3.5%, 12/15/2024, J=J_{\text{max}}=3319
Looking at Data: Bearingpoint

Bearingpoint, 3.5%, 12/15/2024, J=500
Looking at Data: Bearingpoint

Bearingpoint, 3.5%, 12/15/2024, J=5
Looking at Data: Schlumberger
C++ Pricing Code

- Simple Convertible Bond class
- MC Calibration member functions
- Home-grown regression class
- MC Pricing member function (implementing backward induction)
Prices and Model Deltas

Sirius 3.5% 6/1/2008 on 8/9/06, S0=$3.88, conv=72.46
Prices and Model Rhos

Sirius 3.5% 6/1/2008 on 8/9/06, S0=$3.88, conv=72.46

Rho = 0.0691
Prices and Model Vegas

Sirius 3.5% 6/1/2008 on 8/9/06, S0=$3.88, conv=72.46

Vega = 0.125
(Kynex) 0.0993
If recall takes place at time $t = j \Delta t$, holder has $\delta \Delta t$ to convert. So replace $U_j$ by

$$\sup_{j \leq \tau \leq j + \tau} \mathbb{E}\{U_\tau\}$$

i.e. replace the upper reward $\{U_j\}_j$ by the value of an American like option

- Implementation: **Straightforward**
- Computing time: **Prohibitive**
- Easy upper bound by martingale methods (duality)
**Convertible Bonds**

**Proposed Monte Carlo Approach**

**Dynkin Games**

**What Has Been Done**

**Duality/Pointwise Approach**

**American Options**

\[
\sup_{\tau} \mathbb{E}\{X_\tau\} = \inf_{M=\{M_j\}_j} \mathbb{E}\{\sup_j [X_j - M_j]\}
\]

- Davis-Karatzsas
- Haug-Kogan
- Andersen-Broadie, Rogers

**Dynkin Games / Israeli Options**

There exists a martingale \( M = \{M_n\}_n \) s.t.

\[
V_0 = \sup_i \inf_j [R(i, j) - M^*_i \wedge j] = \inf_j \sup_i [R(i, j) - M^*_i \wedge j] \quad \text{a.s.}
\]

- Kühn - Kyprianou - Schaik
American Options

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- Kühn - Kyprianou - Schaik
What Remains to be Done

- Developing Importance Sampling methods for Variance Reduction
- Adding more stochastic factors
  - interest rate (long dated bonds with borrow fees)
  - stochastic volatility (e.g. Heston model)
  - other underliers (stocks, indexes, baskets) for exchange conversions.

  **EASY**, only problem: Regression becomes multivariate !!!

- Identification of the optimal conversion time as the first crossing time of an Exercise Boundary (support of $S(\tau)$)
- Identification of the optimal call time and statistical analysis of the CALL LAG
- Exogeneous modelling of tax effects and Fundamental Changes
- Developing and Implementing Pathwise approach
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