Measuring the impact of jumps on multivariate price processes using multipower variation

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1 Introduction

Review the econometrics of non-parametric estimation of the components of the variation of asset prices.

Paradigm shift in volatility measurement and jump finding.

Econometrics + data + arbitrage free financial economics theory.

Deep impacts on the econometrics of asset allocation, option pricing and risk management.

Probability theory, financial economics, mathematical finance, statistics, econometrics.

1.1 Frictionless model. $\mathcal{BSM}$, jumps and QV

Log-price at time t, ($\mathcal{SM}$) living on some filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ obeys

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u + J_t, \tag{1}$$

where a are predictable drifts, σ are càdlàg, W is a BM, J is a jump process.

$Y - J$ called a Brownian semimartingale ($\mathcal{BSM}$).

Canonical model in the finance theory of continuous sample path processes.

Quadratic variation (QV) process of Y is (EX-POST)

$$[Y]_t = \mathrm{p}-\lim_{n\to\infty} \sum_{j=1}^{t_j \leq t} \left(Y_{t_j} - Y_{t_{j-1}}\right)^2, \tag{2}$$

for any sequence of partitions $0 = t_0 < t_1 < \ldots < t_n = T$ with $\sup_j\{t_{j+1} - t_j\} \to 0$ for $n \to \infty$. Well known that if

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u + J_t$$

then

$$[Y] = \int_0^t \sigma_u^2 \mathrm{d}u + \sum_{u \leq t} (\Delta J_u)^2 .$$

If $Y \in \mathcal{BSM}$

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u$$

writing $\mathcal{F}_t$ as the natural filtration,

$$\mathrm{d}Y_t|\mathcal{F}_t \sim N\left(a_t \mathrm{d}t, \sigma_t^2 \mathrm{d}t\right), \quad [Y]_t = \int_0^t \mathrm{Var}\left(\mathrm{d}Y_u|\mathcal{F}_u\right) \mathrm{d}u. \tag{3}$$

More

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u + J_t$$

$$\mathrm{Var}\left(\mathrm{d}Y_t|\mathcal{F}_t\right) = \sigma_t^2 \mathrm{d}t + \mathrm{Var}\left(\mathrm{d}J_t|\mathcal{F}_t\right) \neq \mathrm{d}[Y]_t.$$

QV process aggregates the components of the variation of prices and so is not sufficient to learn the integrated variance process.

Bipower variation (BPV) process of Barndorff-Nielsen and Shephard (2004b) looks inside $[Y]$

$$\{Y\}_t = \text{p}-\lim_{\delta \downarrow 0} \sum_{j=1}^{\lfloor t/\delta \rfloor} \left| Y_{\delta(j-1)} - Y_{\delta(j-2)} \right| \left| Y_{\delta j} - Y_{\delta(j-1)} \right|. \tag{4}$$

Then

$$\mu_1^{-2} \{Y\}_t = \int_0^t \sigma_u^2 \mathrm{d}u,$$

where $\mu_r = \mathrm{E} |U|^r$, $U \sim N(0,1)$ and $r > 0$.

Multipower variation.

$$\{Y\}_t^{[2/p]} = \text{p}-\lim_{\delta\downarrow 0} \sum_{j=1}^{\lfloor t/\delta \rfloor} \prod_{k=0}^{p-1} \left| Y_{\delta(j-k)} - Y_{\delta(j-k-1)} \right|^{2/p}, \quad p = 1, 2, \ldots.$$

When $p > 1$ then this estimator is robust to jumps.

Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2006)

$$\text{p}-\lim_{\delta\downarrow 0} \sum_{j=1}^{\lfloor t/\delta \rfloor} \prod_{k=0}^{p-1} g_k(Y_{\delta(j-k)} - Y_{\delta(j-k-1)}).$$

Extensions include: Jacod(2007, SPA) & Kinnebrouck and Podolskij (2007, SPA).

1.2 Realised QV & BPV

The QV process can be estimated in many different ways. The realised QV estimator

$$[Y_\delta]_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left(Y_{j\delta} - Y_{(j-1)\delta}\right) \left(Y_{j\delta} - Y_{(j-1)\delta}\right)',$$

where $\delta > 0$. Consistent as $\delta \downarrow 0$, but frictions. Likewise

$$\{Y_\delta\}_t = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left|Y_{\delta(j-1)} - Y_{\delta(j-2)}\right| \left|Y_{\delta j} - Y_{\delta(j-1)}\right|. \tag{5}$$

Define the daily QV

$$V_i = [Y]_i - [Y]_{(i-1)}\,, \quad i = 1, 2, ...$$

estimated by the realised daily QV

$$\widehat{V}_i = [Y_\delta]_i - [Y_\delta]_{(i-1)}\,, \quad i = 1, 2, ...$$

Realised volatility has a long history. It appears in Rosenberg (1972), Merton (1980), Schwert (1989) and Schwert (1998).

Of course, in probability theory QV was discussed as early as Wiener (1924).

Closer connection between realised QV and QV, and its use for econometric purposes, was made in Comte and Renault (1998), Barndorff-Nielsen and Shephard (2001) and Andersen, Bollerslev, Diebold, and Labys (2001).

Substantial literature on writing derivatives on realised volatility. Neuberger (1990), Carr and Madan (1998), Demeterfi, Derman, Kamal, and Zou (1999), Carr and Lewis (2004). Brockhaus and Long (1999), Javaheri, Wilmott, and Haug (2002), Howison, Rafailidis, and Rasmussen (2004), Carr, Geman, Madan, and Yor (2005), Carr and Lee (2003). See also the overview of Branger and Schlag (2005).

Likewise the realised BPV process

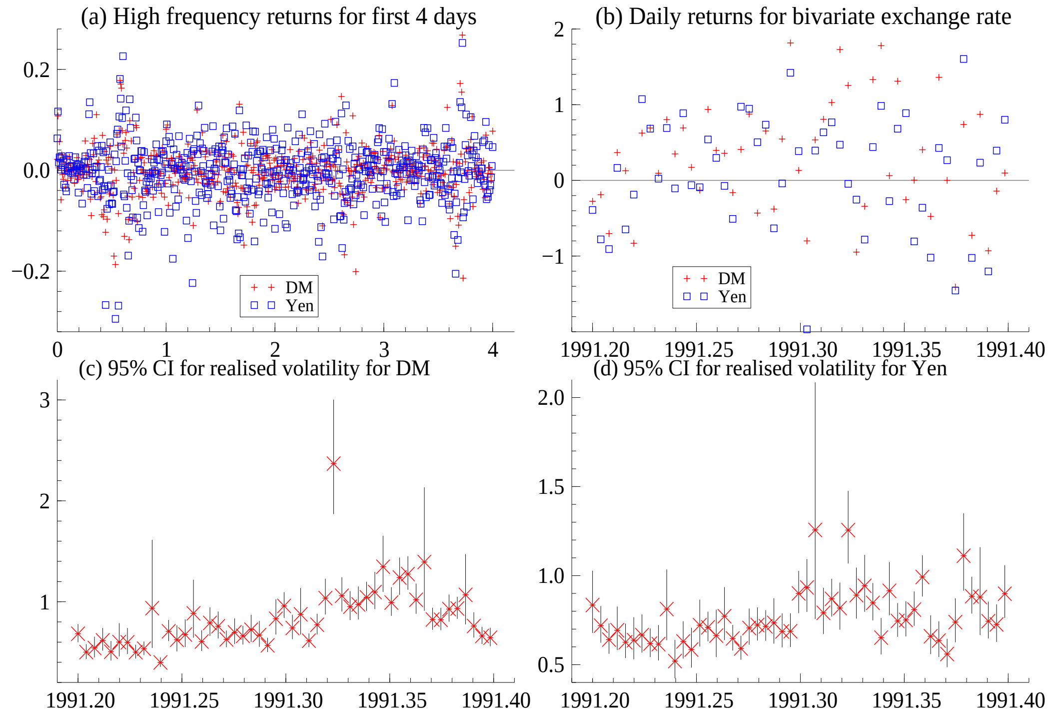
$$\widehat{B}_i = \mu_1^{-2} \left\{ \{Y_\delta\}_i - \{Y_\delta\}_{(i-1)} \right\}, \quad i = 1, 2, ...$$

which estimates

$$B_i = \int_{(i-1)}^{i} \sigma_u^2 \mathrm{d}u, \quad i = 1, 2, ...$$

1.3 Empirical illustrations: measurement

Bivariate series in question records the number of German Deutsche Mark a single US Dollar buys (written Y^1) and Japanese Yen/Dollar series (written Y^2). It starts on February 4th 1991 and covers the next 50 trading days. Inference based on 10 minute returns.


(a) High frequency returns for first 4 days
0.2
0.0
−0.2
0
1
2
3
4
DM
Yen
(b) Daily returns for bivariate exchange rate
2
1
0
−1
1991.20
1991.25
1991.30
1991.35
1991.40
DM
Yen
(c) 95% CI for realised volatility for DM
3
2
1
1991.20
1991.25
1991.30
1991.35
1991.40
(d) 95% CI for realised volatility for Yen
2.0
1.5
1.0
0.5
1991.20
1991.25
1991.30
1991.35
1991.40

2 Measurement error when $Y \in \mathcal{BSM}$

CLT for $[Y_\delta]_t$ can be discretise the result to produce the desired results for $\widehat{V}_i$. Univariate results stated only. These results was developed in a series of papers by Jacod (1994), Jacod and Protter (1998), Barndorff-Nielsen and Shephard (2002).

$$\delta^{-1/2}\left([Y_\delta]_t - [Y]_t\right) \xrightarrow{L} MN\left(0, 2t\int_0^t \sigma_u^4 \mathrm{d}u\right), \tag{6}$$

where MN denotes a mixed Gaussian distribution. Barndorff-Nielsen and Shephard (2002) named $\int_0^t \sigma_u^4 \mathrm{d}u$ *integrated quarticity*.

$\int_0^t \sigma_u^4 \mathrm{d}u$ can be consistently estimated using $(1/3)\left\{Y_\delta\right\}_t^{[4]}$ where realised quarticity

$$\left\{Y_\delta\right\}_t^{[4]} = \delta^{-1} \sum_{j=1}^{\lfloor t/\delta \rfloor} \left(Y_{j\delta} - Y_{(j-1)\delta}\right)^4. \tag{7}$$

We get the Barndorff-Nielsen and Shephard (2002) results

$$\frac{\delta^{-1/2}\left([Y_\delta]_t - [Y]_t\right)}{\sqrt{t\frac{2}{3}\left\{Y_\delta\right\}_t^{[4]}}} \xrightarrow{L} N(0,1), \tag{8}$$

$$\frac{\delta^{-1/2}\left(\log[Y_\delta]_t - \log[Y]_t\right)}{\sqrt{\frac{2}{3}\frac{t\left\{Y_\delta\right\}_t^{[4]}}{\left([Y_\delta]_t\right)^2}}} \xrightarrow{L} N(0,1). \tag{9}$$

2.1 Jumps + bipower variation

Recall when Y is a $\mathcal{BSM}$ plus jump process then

$$\mu_1^{-2}\{Y\}_t = \int_0^t \Sigma_u \mathrm{d}u,$$

Under the assumption that there are no jumps, then

$$\delta^{-1/2}\begin{pmatrix} \mu_1^{-2}\{Y_\delta\}_t - \mu_1^{-2}\{Y\}_t \\ [Y_\delta]_t - [Y]_t \end{pmatrix}$$

$$\xrightarrow{L} MN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (2+\vartheta) & 2 \\ 2 & 2 \end{pmatrix}\int_0^t \sigma_u^4 \mathrm{d}u\right),$$

$$\vartheta = \left(\pi^2/4\right) + \pi - 5 \simeq 0.6090.$$

Const	Realised terms $\widehat{V}_{i-1}$	$\widehat{B}_{i-1}$	Standard GARCH terms $(Y_{i-1}-Y_{i-2})^2$	h_{i-1}	$\log L$
0.008			0.053	0.930	-2552.10
(0.003)			(0.010)	(0.013)	
0.017	-0.115	0.253	0.019	0.842	-2533.89
(0.009)	(0.039)	(0.076)	(0.019)	(0.052)	

Table 1: GARCH model for *100* $(Y_i - Y_{i-1})$ *on the DM/Dollar series. Using lagged squared returns* $(Y_{i-1} - Y_{i-2})^2$, *and lagged conditional variance* h_{i-1}. *Gaussian quasi-likelihood is used. Robust s.e.s are reported.*

Consequently Barndorff-Nielsen and Shephard (2006) used

$$\frac{\delta^{-1/2}\left([Y_\delta]_t - \mu_1^{-2}\{Y_\delta\}_t\right)}{\sqrt{\vartheta \int_0^t \sigma_u^4 \mathrm{d}u}} \xrightarrow{L} N\left(0,1\right), \tag{10}$$

as the basis of a test of the null of no jumps.

2.2 Multipower variation

Estimate $\int_0^t \sigma_u^4 \mathrm{d}u$ robustly to jumps? Realised multipower variation (MPV) measure (Barndorff-Nielsen and Shephard (2006)). e.g.

$$\begin{aligned} \{Y_\delta\}_t^{[1,1,1,1]} &= \delta^{-1} \sum_{j=1}^{\lfloor t/\delta \rfloor} \left\{ \prod_{i=1}^{4} \left| Y_{\delta(j-i)} - Y_{\delta(j-1-i)} \right| \right\} \\ &\xrightarrow{p} \mu_1^4 \int_0^t \sigma_u^4 \mathrm{d}u, \end{aligned}$$

So $\mu_1^{-4} \{Y_\delta\}_t^{[1,1,1,1]}$ estimates $\int_0^t \sigma_u^4 \mathrm{d}u$ consistently in the presence of jumps.

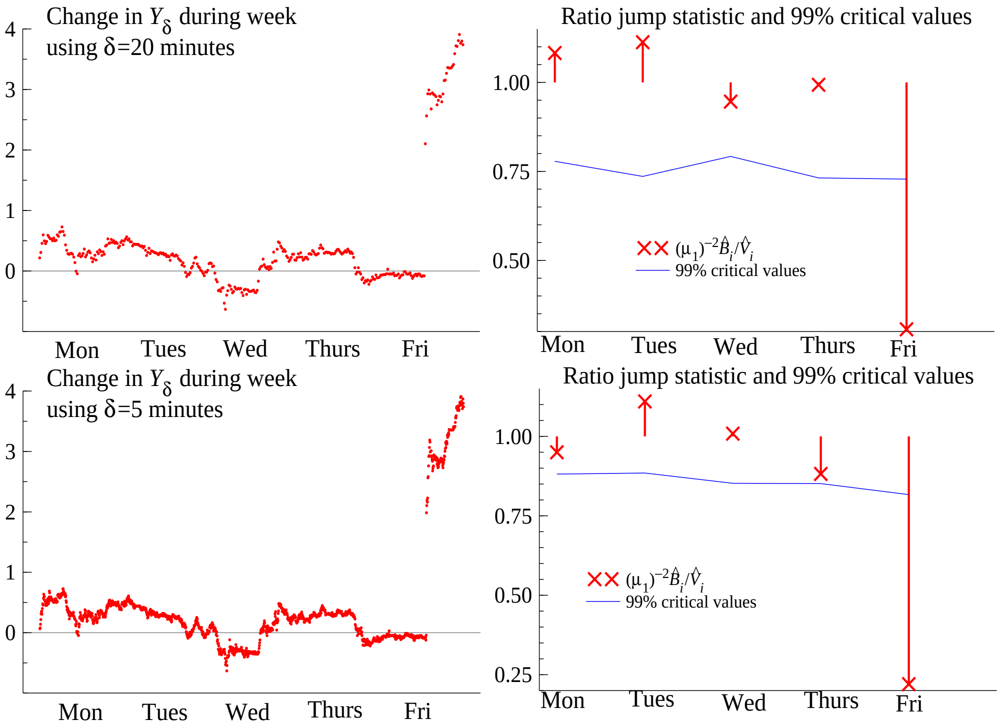
The test is conditionally consistent and has asymptotically the correct size. Extensive small sample studies are reported in Huang and Tauchen (2005), who favour ratio versions of the statistic like

$$\frac{\delta^{-1/2}\left(\frac{\mu_1^{-2}\{Y_\delta\}_t}{[Y_\delta]_t}-1\right)}{\sqrt{\vartheta\frac{\{Y_\delta\}_t^{[1,1,1,1]}}{\left(\{Y_\delta\}_t\right)^2}}}\overset{L}{\to}N(0,1),$$

which has pretty reasonable finite sample properties. They also show that this test tends to under reject the null of no jumps in the presence of some forms of market frictions.

Carry out jump testing on separate days or weeks. Asymptotically independent under the null hypothesis.

DM/Dollar rate. Our focus will mostly be on Friday January 15th 1988, although we will also give results for neighbouring days to provide some context. In Figure 2.2 we plot 100 times the discretised Y_δ, so a one unit uptick represents a 1% change, for a variety of values of $n = 1/\delta$, as well as giving the ratio jump statistics $\widehat{B}_i/\widehat{V}_i$ with their corresponding 99% critical values.


Change in Y_δ during week
using δ=20 minutes
Ratio jump statistic and 99% critical values
$(\mu_1)^{-2}\hat{B}_i/\hat{V}_i$
99% critical values
Mon
Tues
Wed
Thurs
Fri
Change in Y_δ during week
using δ=5 minutes
Ratio jump statistic and 99% critical values
$(\mu_1)^{-2}\hat{B}_i/\hat{V}_i$
99% critical values

2.3 Research problems

- distribution theory under jumps
- multivariate case
- noise

Under jumps for the tripower variation,

$$\{Y_\delta\}_t^{[2/3]} = \sum_{j=1}^{\lfloor t/\delta \rfloor} \left\{ \prod_{i=1}^{3} \left| Y_{\delta(j-i)} - Y_{\delta(j-1-i)} \right|^{2/3} \right\}$$

we have the following result with Barndorff-Nielsen and Veraart

$$\delta^{-1/2}\begin{pmatrix} \mu_{2/3}^{-3}\{Y_\delta\}_1^{[2/3]} - \int_0^1 \sigma_u^2 \mathrm{d}u \\ [Y_\delta]_t - [Y]_t \end{pmatrix}$$

$$\xrightarrow{L} MN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (2+c) & 2 \\ 2 & 2 \end{pmatrix}\int_0^t \sigma_u^4 \mathrm{d}u + 4\sum_{u<1}\begin{pmatrix} 0 & 0 \\ 0 & (\Delta Y_u)^2\sigma_u^2 \end{pmatrix}\right)$$

Thus the asymptotic distribution, not just the probability limit, is invariant to jumps for tripower variation.

Allows one to make inference on $\int_0^1 \sigma_u^2 \mathrm{d}u$ robustly in the presence of jumps.

2.4 Multivariate case

Multivariate process

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u + J_t, \tag{11}$$

where a are predictable drifts, σ are càdlàg, W is a vector BM, J is a jump process.

$$[Y] = \int_0^t \sigma_u^2 \mathrm{d}u + \sum (\Delta Y_u)^2 .$$

Bipower variation (BPV) process

$$\left\{Y^{l}\right\}_{t}=\mathrm{p}-\lim_{\delta\downarrow 0}\sum_{j=1}^{\lfloor t/\delta\rfloor}\left|Y^{l}_{\delta(j-1)}-Y^{l}_{\delta(j-2)}\right|\left|Y^{l}_{\delta j}-Y^{l}_{\delta(j-1)}\right|. \tag{12}$$

The $p\times p$ matrix BPV process $\{Y\}$ has l,k-th element

$$\left\{Y^{l},Y^{k}\right\}=\frac{1}{4}\left(\left\{Y^{l}+Y^{k}\right\}-\left\{Y^{l}-Y^{k}\right\}\right),\quad l,k,=1,2,...,p. \tag{13}$$

$$\mu_{1}^{-2}\left\{Y\right\}_{t}=\int_{0}^{t}\Sigma_{u}\mathrm{d}u,$$

where $\mu_r=\mathrm{E}\left|U\right|^{r}$, $U\sim N(0,1)$ and $r>0$.

Can develop tripower version of this too, whose asymptotic distribution is not effected by jumps. The implication is that we can test for common multivariate jumps allowing for univariate jumps. This is work in progress and we will report that another time.

2.5 Noise

Noise is important at high frequency. Much progress for realised variance

- realised kernels, Barndorff-Nielsen, Hansen, Lunde, and Shephard (2006)
- multiscale estimators, Zhang (2006) and Zhang, Mykland, and Aït-Sahalia (2005)

Main stories: slow rates of convergence, but still mixed Gaussian limit theory. Improvements in practice over using 5 minute return data.

Multipower is harder as we do not work with squared observations.

Approach by Mark Podolskij and coworkers in various papers has been to average returns over small periods of time before computing multi-power variation statistics. Then bias correct the resulting estimator. Show this has good theoretical properties, but Monte Carlos have so far not been very impressive.

3 Conclusions

QV is perhaps the key object in the financial econometrics of price processes

A lot of recent and exciting work in this area.

Much is semiparametric.

Uses high frequency data.

Paradigm shift in volatility forecasting

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