

# **Sensitivity analysis of utility based prices and risk-tolerance wealth processes**

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Based on a paper with Mihai Sirbu from Columbia University

# Outline

The prices of **non replicable** derivative securities depend on many factors:

1. risk-preferences of the investor:

(a) reference probability measure  $\mathbb{P}$

(b) utility function  $U = U(x)$

2. current portfolio of the investor

3. trading volume in the derivatives

**Goal:** study the **dependence** of prices on trading volume.

# Model of a financial market

There are  $d + 1$  traded or liquid assets:

1. a **savings account** with zero interest rate.
2.  $d$  **stocks**. The price process  $S$  of the stocks is a semimartingale on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ .

$\mathcal{Q}$  : the family of local martingale measures for  $S$ .

**Assumption** (*No Arbitrage*)

$$\mathcal{Q} \neq \emptyset$$

# Contingent claims

Consider a family of  $m$  non-traded or illiquid European contingent claims with

1. maturity  $T$
2. payment functions  $f = (f_i)_{1 \leq i \leq m}$ .

**Assumption** *No nonzero portfolio of  $f$  is replicable:*

$$\langle q, f \rangle = \sum_{i=1}^m q_i f_i \text{ is replicable} \Leftrightarrow q = 0$$

# Pricing problem

**Question** What is the (marginal) price  $p = (p_i)_{1 \leq i \leq m}$  of the contingent claims  $f$  ?

**Intuitive Definition** The marginal price  $p$  for the contingent claims  $f$  is the **threshold** such that given the chance to buy or sell at  $p^{trade}$  an investor will

1. buy at  $p^{trade} < p$
2. sell at  $p^{trade} > p$
3. do nothing at  $p^{trade} = p$

# Economic agent or investor

Consider an investor with the portfolio  $(x, q)$ , whose preferences are modeled by a **utility function**  $U$ :

1.  $U : (0, \infty) \rightarrow \mathbf{R}$ , strictly increasing and strictly concave
2. The Inada conditions hold true:

$$U'(0) = \infty \quad U'(\infty) = 0$$

# Problem of optimal investment

The goal of the investor is to maximize **the expected utility of terminal wealth**:

$$u(x, q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + \langle q, f \rangle)],$$

where  $\mathcal{X}(x)$  is the set of strategies with initial wealth  $x$ .

**Order structure:** a portfolio  $(x, q)$  is **better** than a portfolio  $(x', q')$  if  $u(x, q) \geq u(x', q')$ .

# Marginal utility based price

**Definition** A marginal utility based price for the claims  $f$  given the portfolio  $(x, q)$  is a vector  $p(x, q)$  such that

$$u(x, q) \geq u(x', q')$$

for any pair  $(x', q')$  satisfying

$$x + \langle q, p(x, q) \rangle = x' + \langle q', p(x, q) \rangle.$$

In other words, given the portfolio  $(x, q)$  the investor **will not trade** the options at  $p(x, q)$ .

# Computation of $p(x) = p(x, 0)$

Define the conjugate function

$$V(y) = \max_{x>0} [U(x) - xy], \quad y > 0.$$

and consider the following **dual** optimization problem:

$$v(y) = \inf_{Q \in \mathcal{Q}} \mathbb{E} \left[ V \left( y \left( \frac{dQ}{dP} \right) \right) \right], \quad y > 0$$

$\mathcal{Q}(y)$  : the **minimal martingale measure** for  $y$  .

# Computation of $p(x) = p(x, 0)$

Mark Davis gave heuristic arguments to show that if  $y$  corresponds to  $x$  in the sense that

$$x = -v'(y) \Leftrightarrow y = u'(x)$$

then

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

The precise mathematical results are given in a joint paper with Julien Hugonnier and Walter Schachermayer.

# Computation of $p(x) = p(x, 0)$

**Theorem (Hugonnier, K., Schachermayer)** Let  $x > 0$ ,  $y = u'(x)$  and  $X$  be a non-negative wealth process. The following conditions are equivalent:

1.  $p(x)$  is unique for any  $f$  such that

$$|f| \leq K(1 + X_T) \text{ for some } K > 0$$

2.  $\mathbb{Q}(y)$  exists and  $X$  is a martingale under  $\mathbb{Q}(y)$ .

Moreover, in this case  $p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f]$ .

# Trading problem

Assume that the investor can trade the claims at the initial time at the price  $p^{trade}$ .

**Question** What quantity  $q = q(p^{trade})$  the investor should trade (buy or sell) at the price  $p^{trade}$  ?

Using the marginal utility based prices  $p(x, q)$  we can compute the optimal quantity from the “equilibrium” condition:

$$p^{trade} = p(x - qp^{trade}, q)$$

# Trading problem

Qualitatively, there are two different cases:

1. If  $f$  is **replicable**, then

$$q(p^{trade}) = \begin{cases} \text{any} & p^{trade} = p(x) \\ \infty, & p^{trade} \neq p(x) \end{cases}$$

2. else if  $f$  is **not replicable**, then

$$q(p^{trade}) = \begin{cases} 0 & p^{trade} = p(x) \\ \text{finite} & p^{trade} \neq p(x) \end{cases}$$

# Sensitivity analysis of utility based prices

**Main difficulty** :  $p(x, q)$  is hard to compute except for the case  $q = 0$ .

**Linear approximation** for “small”  $\Delta x$  and  $q$  :

$$p(x + \Delta x, q) \approx p(x) + p'(x)\Delta x + D(x)q,$$

where  $p'(x)$  is the derivative of  $p(x)$  and

$$D^{ij}(x) = \frac{\partial p^i}{\partial q^j}(x, 0), \quad 1 \leq i, j \leq m.$$

# Quantitative questions

**Question (Quantitative)** How to compute  $p'(x)$  and  $D(x)$  ?

**Closely related publications:**

**J. Kallsen (02)** : formula for  $D(x)$  for general semimartingale model but in a different framework of local utility maximization.

**V. Henderson (02)** : formula for  $D(x)$  in the case of a Black-Scholes type model with basis risk and for power utility functions.

# Qualitative questions

**Question (Qualitative)** When the following (desirable) properties hold true for **any** family of contingent claims  $f$  ?

1. The marginal utility based price  $p(x) = p(x, 0)$  **does not depend** (locally) on  $x$  , that is,

$$p'(x) = 0$$

2. The sensitivity matrix  $D(x)$  has **full rank**
3. The sensitivity matrix  $D(x)$  is **symmetric**

# Qualitative questions

4. The sensitivity matrix  $D(x)$  is **negative semi-definite**:  
 $\langle q, D(x)q \rangle \leq 0$ .
5. **Stability** of the linear approximation: for any  $p^{trade}$  the linear approximation to the “equilibrium” equation:

$$p^{trade} = p(x - qp^{trade}, q)$$

that is,

$$p^{trade} \approx p(x) - p'(x)qp^{trade} + D(x)q$$

has the “correct” solution.

# Risk-tolerance wealth process

**Definition (K., Sirbu)** A maximal wealth process  $R(x)$  is called the **risk-tolerance wealth process** if

$$R_T(x) = -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))},$$

where  $\widehat{X}(x)$  is the optimal solution of

$$u(x) := u(x, 0) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

# Risk-tolerance wealth process

Some properties of  $R(x)$  (if it exists):

1. Initial value:

$$R_0(x) = -\frac{u'(x)}{u''(x)}.$$

2. Derivative of optimal wealth strategy:

$$\frac{R(x)}{R_0(x)} = X'(x) := \lim_{\Delta x \rightarrow 0} \frac{\widehat{X}(x + \Delta x) - \widehat{X}(x)}{\Delta x}.$$

# Main qualitative result

Recall  $p(x + \Delta x, q) \approx p(x) + p'(x)\Delta x + D(x)q$ .

**Theorem (K., Sirbu)** *The following assertions are equivalent:*

1. *The risk-tolerance wealth process  $R(x)$  exists.*
2.  *$p'(x) = 0$  for any  $f$ .*
3.  *$D(x)$  is symmetric for any  $f$ .*
4.  *$D(x)$  has full rank for any (non-replicable)  $f$ .*
5.  *$D(x)$  is negative semidefinite for any  $f$ .*

# Existence of $R(x)$

Recall that  $\mathbb{Q}(y)$  is the solution of

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E} \left[ V \left( y \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right) \right], \quad y > 0$$

where  $V$  is the convex conjugate of  $U$ .

**Theorem (K., Sirbu)** *The following assertions are equivalent:*

1.  $R(x)$  exists for any  $x > 0$ .
2.  $\mathbb{Q}(y) = \hat{\mathbb{Q}}$  (same for all  $y$ !).

# Second order stochastic dominance

**Definition** If  $\xi$  and  $\eta$  are nonnegative random variables, then  $\xi \succeq_2 \eta$  if

$$\int_0^t \mathbb{P}(\xi \geq x) dx \geq \int_0^t \mathbb{P}(\eta \geq x) dx, \quad t \geq 0.$$

We have that  $\xi \succeq_2 \eta$  iff

$$\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$$

for any *convex and decreasing* function  $W$ .

# Existence of $R(x)$

**Case 1:** a utility function  $U$  is arbitrary.

**Theorem (K., Sirbu)** *The following assertions are equivalent:*

1.  $R(x)$  exists for any  $x > 0$  and any utility function  $U$ .
2. There exists a unique  $\hat{Q} \in \mathcal{Q}$  such that

$$\frac{d\hat{Q}}{dP} \succeq_2 \frac{dQ}{dP} \quad \forall Q \in \mathcal{Q}.$$

# Existence of $R(x)$

**Case 2:** a financial model is arbitrary.

**Theorem (K., Sirbu)** *The following assertions are equivalent:*

1.  $R(x)$  exists for any  $x > 0$  and any financial model.
2. The utility function  $U$  is
  - (a) the power utility:  $U(x) = (x^\alpha - 1)/\alpha$ ,  $\alpha < 1$ ,  
if  $x \in (0, \infty)$ ;
  - (b) the exponential utility:  $U(x) = -\exp(-\gamma x)$ ,  
 $\gamma > 0$ , if  $x \in (-\infty, \infty)$ .

# Computation of $D(x)$

We choose

$$R(x)/R_0(x) = X'(x)$$

as a **numéraire** and denote

$f^R = f R_0(x)/R(x)$  : discounted contingent claims

$X^R = X R_0(x)/R(x)$  : discounted wealth processes

$\mathbb{Q}^R$  : the martingale measure for  $X^R$  :

$$\frac{d\mathbb{Q}^R}{d\hat{\mathbb{Q}}} = \frac{R_T(x)}{R_0(x)}$$

# Computation of $D(x)$

Consider the Kunita-Watanabe decomposition:

$$P_t^R = \mathbb{E}_{\mathbb{Q}^R} [f^R | \mathcal{F}_t] = M_t + N_t, \quad N_0 = 0,$$

where

1.  $M$  is  $R(x)/R_0(x)$ -discounted wealth process.

Interpretation: **hedging process**.

2.  $N$  is a martingale under  $\mathbb{Q}^R$  which is orthogonal to all  $R(x)/R_0(x)$ -discounted wealth processes.

Interpretation: **risk process**.

# Computation of $D(x)$

Denote  $a(x) := -xu''(x)/u'(x)$  the relative risk-aversion coefficient of

$$u(x) = \max_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

**Theorem (K., Sirbu)** *Assume that the risk-tolerance wealth process  $R(x)$  exists. Then*

$$D(x) = -\frac{a(x)}{x} \mathbb{E}_{\mathbb{Q}^R} [N_T N_T']$$

# Computation of $D(x)$ in practice

## Inputs:

1.  $\hat{Q}$  . *Already implemented!*
2.  $R(x)/R_0(x)$  . Recall that

$$\frac{R(x)}{R_0(x)} = \lim_{\Delta x \rightarrow 0} \frac{\hat{X}(x + \Delta x) - \hat{X}(x)}{\Delta x}.$$

*Decide what to do with one penny!*

3. Relative risk-aversion coefficient  $\alpha(x)$  . *Deduce from mean-variance preferences.* In any case, this is just a number!

# Model with basis risk

Traded asset :  $dS_t = S_t (\mu dt + \sigma dW_t)$  .

Non traded asset :  $d\tilde{S} = (\tilde{\mu} dt + \tilde{\sigma} d\tilde{W}_t)$

Denote by

$$\rho = \frac{d\tilde{W} dW}{dt}$$

the **correlation** coefficient between  $S$  and  $\tilde{S}$  . In practice, we want to chose  $S$  so that

$$\rho \approx 1.$$

# Model with basis risk

Consider contingent claims  $f = f(\tilde{S})$  whose payoffs are determined by  $\tilde{S}$  (maybe path dependent).

To compute  $D(x)$  assume (as an example) the following choices:

1.  $\hat{\mathbb{Q}}$  is a martingale measure for  $\tilde{S}$ .
2.  $R(x)/R_0(x) = 1$

Then

$$D_{ij}(x) = -\frac{\alpha(x)}{x} (1 - \rho^2) \text{Cov}_{\hat{\mathbb{Q}}}(f_i, f_j).$$

# Existence of $u''(x)$

Recall that

$$u(x) = \max_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

1. Does  $u''(x)$  exist? Is  $u''(x)$  strictly positive?
2. Does

$$X'(x) = \lim_{\Delta x \rightarrow 0} \frac{X(x + \Delta x) - X(x)}{\Delta x}$$

exist?

# Assumption on the utility function

**Assumption** *There are strictly positive constants  $c_1$  and  $c_2$  such that*

$$c_1 < -\frac{xU''(x)}{U'(x)} < c_2, \quad x > 0.$$

# Sigma-bounded semimartingales

**Definition** A semimartingale  $R$  is called **sigma-bounded** if there is  $H > 0$  such that the stochastic integral  $\int H dR$  is bounded.

**Remark** If  $R$  is locally bounded, then  $R$  is sigma-bounded. The reverse assertion does not hold true.

# Assumption on the (jumps of) price processes

**Assumption** *The price process of the traded securities discounted by  $X(x)$ , that is,*

$$S^{X(x)} = \left( \frac{1}{X(x)}, \frac{S}{X(x)} \right)$$

*is sigma-bounded.*

**Remark** The above assumption is useful only if we can **explicitly** compute the optimal investment strategy.

# Assumption on the (jumps of) price processes

The last assumption is satisfied in the following cases:

1.  $S$  is continuous
2. The financial model can be "completed".
3. If purely discontinuous martingales have finite-dimensional basis (as stochastic integrals).

**Remark** There are complete financial models, where the "locally bounded analog" of the assumption is not satisfied.

# Assumption on the (jumps of) price processes

**Theorem (K., Sirbu)** *Assume that there is a finite-dimensional local martingale  $M$  such that any bounded purely discontinuous martingale  $N$  is given by*

$$N = N_0 + \int H dM.$$

*Then every semimartingale on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  is sigma-bounded.*

**Remark** The condition of the theorem is invariant under an equivalent change of the reference probability measure.

# Main result

**Theorem (K., Sirbu)** *Let both assumptions above be satisfied. Then  $u''(x)$  and  $X'(x)$  exist for any  $x > 0$  and*

$$c_1 < -\frac{xu''(x)}{u'(x)} < c_2, \quad x > 0.$$

*where  $c_1$  and  $c_2$  are the same constants as in the assumption on  $U$ .*

**Remark** Both assumptions are **essential** for the assertion of the theorem to hold true.

# Summary

- For non replicable contingent claims prices depend on the trading volume.
- The following conditions are equivalent:
  - Approximate utility based prices have nice qualitative properties
  - Risk-tolerance wealth processes exist.
- We need to solve the mean-variance hedging problem, where the risk-tolerance wealth process plays the role of the numéraire.