Debt Maturity and Endogenous Default within Optimal Stopping Theory

by

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for

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Credit modelling: the structural approach.

▷ Attempt to model the evolution of the assets of a firm
  • Pricing debt and equity as contingent claims.
  • Default is endogenous.
  • Estimate credit spreads.


▷ Conflicting results for short term debt.
  • structural models underestimate spreads and/or default risks for short horizon
  • Eom, Helwege and Huang (2004), RFS: Leland papers overestimate spreads and default risks and substantially at short maturities.

▷ Possible explanation:
  • literature restricts the set of bankruptcy policies to hitting times.
  • underestimation of the value of equity.
Structure of the firm.

▷ The value of the firm’s asset evolves as:

\[ V_t = V_0 \exp(X_t) \]

where \( X \) is a Lévy process with no upward jumps.

▷ The firm is partly financed by debt that is constantly retired and reissued in the following way:

- issue debt with face value \( pdt \) and maturity profile \( \phi \) where \( \phi \) is the density function of a positive random variable.
- Specification: \( \phi(t) = me^{-mt} \).
- At time 0, the amount that will be paid at time \( s \) is

\[
p \int_s^\infty \phi(t) \, dt \, ds
\]

- Face value of the total debt is constant and is worth \( F = \frac{p}{m} \).

▷ Bankruptcy cost: \( \alpha \) and tax rebate on coupon payment: \( \theta \).
Value of the firm’s stake.

▷ Default is endogenously defined by equity-holders who set the time-to-bankruptcy $\tau$ so as to maximize the value of their equity.

▷ Debt value when $V_0 = V$ is

$$D(V, \tau) = \frac{C + mF}{m + r} - \mathbb{E} \left[ e^{-(r+m)\tau} \left( \frac{C + mF}{m + r} - (1 - \alpha)V_\tau^V \right) \right]$$

▷ Firm value when $V_0 = V$ is

$$v(V, \tau) = V + \frac{\theta C}{r} - \mathbb{E} \left[ e^{-r\tau} \left( \frac{\theta C}{r} + \alpha V_\tau^V \right) \right].$$

▷ Equity value when $V_0 = V$ is

$$E(V) = \sup_{\tau \in T_0, \infty} (v(V, \tau) - D(V, \tau)) \quad (1)$$

$$= V + \frac{\theta C}{r} - \frac{C + mF}{r + m} + P(V)$$

where

$$P(V) = \sup_{\tau \in T_0, \infty} \mathbb{E} \left[ e^{-r\tau} (\beta(\tau) - \gamma(\tau)V_\tau^V) \right]. \quad (2)$$

with

$$\beta(t) = \frac{C + mF}{r + m} e^{-mt} - \frac{\theta C}{r} \quad \text{and} \quad \gamma(t) = \alpha + (1 - \alpha)e^{-mt}.$$
Leland’s Approximation.

▷ Restrict bankruptcy policies to hitting times
\[ \tau_B = \inf \{ t : V_t \leq V_B \} \]
and consider
\[
\begin{align*}
\max_{\tau_B} (E(V, \tau_B) = v(V, \tau_B) - D(V, \tau_B)), \\
v(V, \tau_B) - D(V, \tau_B) \geq 0.
\end{align*}
\]

▷ In a large class of models, we have an explicit computation of \( E(V, \tau_B) \),
\[
V + \frac{\theta C}{r} - \frac{C + mF}{r + m} + \mathbb{E} \left[ e^{-r\tau_B} \left( \beta(\tau_B) - \gamma(\tau_B)V_{\tau_B} \right) \right],
\]

▷ Smooth-fit principle

• Solve
\[
\frac{\partial E(., \tau_B)}{\partial V}(V_B, V_B) \geq 0.
\]

• optimal threshold
\[
V_B = \frac{y \frac{\theta C}{r} - y(m) \frac{C + mF}{r + m}}{1 - y(m) + \alpha(y(m) - y)}.
\]
Solving the optimal problem.

▷ Introduce the process \((P_t)_{t \geq 0}\) defined by
\[
P_t = \text{ess sup}_{\tau \in T_{t,\infty}} \mathbb{E} \left[ e^{-rt} (\beta(\tau) - \gamma(\tau)V_{\tau}) | \mathcal{F}_t \right]. \tag{3}
\]

▷ The process \((P_t)_{t \geq 0}\) is the Snell envelope of the process \((e^{-r t}(\beta(t) - \gamma(t)V_t))_{t \geq 0}\), that is the smallest supermartingale that dominates \((e^{-r t}(\beta(t) - \gamma(t)V_t))_{t \geq 0}\).

▷ The strong Markov property gives the relation
\[
P_t = e^{-r t} P(t, V_t)
\]
where
\[
P(t, V) = \sup_{\tau \in T_{0,\infty}} \mathbb{E} \left[ e^{-r \tau} (\beta(t + \tau) - \gamma(t + \tau)V_{\tau}^V) \right]. \tag{4}
\]

▷ The option value \(P\) defined in (2) satisfies
\[
P(V) = P(0, V)
\]
deserves particular comments. On the one hand, if \(m\) is sufficiently large the put option (2) associated to the irreversible bankruptcy decision has a negative payoff and must be never exercised (▷ Stopping region
\[
S = \{(t, V) \in [0, \infty) \times (0, \infty), P(t, V) = \beta(t) - \gamma(t)V\}
\]
▷ Optimal stopping time
\[
\tau_S(t, V) = \inf\{s \geq 0, (t + s, V_{s}^V) \in S\}.
\]
Solving the optimal problem (2).

▷ The t-sections $S_t$ of the stopping region $S$.

$$S_t = \{ V > 0 \mid P(t, V) = \beta(t) - \gamma(t)V \}.$$

▷ Optimal bankruptcy will occur for $V \in S_0$.

▷ Analytical characterization.

$P(t, .)$ is differentiable and $P$ satisfies almost everywhere

$$\max\left( \frac{\partial P}{\partial t} + AP - rP, (\beta(.) - \gamma(.) - P) \right) = 0$$

▷ Shape of the t-sections.

Let $\tilde{t} = \frac{1}{m} \ln \left( \frac{r(C + mF)}{\theta C(r + m)} \right)$ then,

(i) For $t \geq \tilde{t}$, the t-sections $S_t$ are empty.

(ii) For all $0 \leq t < \tilde{t}$, the t-sections are left connected, that is there exists a strictly positive function $b^*(t)$ such that $S_t = [0, b^*(t)]$.

▷ As a consequence, $E(V) > E(V, \tau_B)$. The Leland approximation underestimates the equity value, to what extent?
Short term maturity

▷ When $m = 0$, the average maturity is infinite and the optimal stopping problem (2) coincides with the Leland solution.

▷ When $m = \infty$, (the average maturity is zero) the value of the put option (2) tends to zero. Therefore, the equity value $E$ satisfies

$$\lim_{m \to \infty} E = V + \frac{\theta C}{r} - F.$$

▷ When $m = \infty$, the Leland equity value is

$$E(V, \tau_B) = V + \frac{\theta C}{r} - F + \mathbb{E} \left[ e^{-r\tau_B} \left( \beta(\tau_B) - \gamma(\tau_B)V^\tau_B \right) \right]$$

$$= V + \frac{\theta C}{r} - F - \left( \alpha \frac{F}{1 - \alpha} + \frac{\theta C}{r} \right) \left( \frac{V(1 - \alpha)}{F} \right)^y$$

▷ When $m = \infty$, the optimal bankruptcy threshold is thus $F - \frac{\theta C}{r}$.

▷ When $m = \infty$, the Leland approximation threshold is $F/(1 - \alpha)$.
Numerical Results

\[ \sigma = 0.2, \ r = 7.5\%, \ \delta = 7\%, \ \alpha = 50\%, \ \theta = 35\% \]
\[ V = 100, \ F = 40 \text{ and } C = 5 \]

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