Debt Maturity and Endogenous Default within Optimal Stopping Theory

by

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Credit modelling: the structural approach.

▷ Attempt to model the evolution of the assets of a firm

- Pricing debt and equity as contingent claims.
- Default is endogenous.
- Estimate credit spreads.

▷ Representative papers: Leland (1994) and Leland and Toft (1996).

▷ Conflicting results for short term debt.

- structural models underestimate spreads and/or default risks for short horizon
- Eom, Helwege and Huang (2004), RFS: Leland papers overestimate spreads and default risks and substantially at short maturities.

▷ Possible explanation:

- literature restricts the set of bankruptcy policies to hitting times.
- underestimation of the value of equity.

Structure of the firm.

▷ The value of the firm's asset evolves as:

$$V_t = V_0 \exp(X_t)$$

where X is a Lévy process with no upward jumps.

▷ The firm is partly financed by debt that is constantly retired and reissued in the following way:

- issue debt with face value pdt and maturity profile ϕ where ϕ is the density function of a positive random variable.
- Specification: $\phi(t) = me^{-mt}$.
- At time 0, the amount that will be paid at time s is

$$p \int_s^\infty \phi(t)\, dtds$$

- Face value of the total debt is constant and is worth $F = \frac{p}{m}$.

▷ Bankruptcy cost: α and tax rebate on coupon payment: θ.

Value of the firm's stake.

▷ Default is endogenously defined by equity-holders who set the time-to-bankruptcy τ so as to maximize the value of their equity.

▷ Debt value when $V_0 = V$ is

$$D(V,\tau) = \frac{C+mF}{m+r} - \mathbb{E}\left[e^{-(r+m)\tau}\left(\frac{C+mF}{m+r} - (1-\alpha)V_\tau^V\right)\right]$$

▷ Firm value when $V_0 = V$ is

$$v(V,\tau) = V + \frac{\theta C}{r} - \mathbb{E}\left[e^{-r\tau}\left(\frac{\theta C}{r} + \alpha V_\tau^V\right)\right].$$

▷ Equity value when $V_0 = V$ is

$$\begin{aligned} E(V) &= \sup_{\tau \in \mathcal{T}_{0,\infty}} \left(v(V,\tau) - D(V,\tau)\right) \qquad (1) \\ &= V + \frac{\theta C}{r} - \frac{C+mF}{r+m} + P(V) \end{aligned}$$

where

$$P(V) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau}(\beta(\tau) - \gamma(\tau)V_\tau^V)\right]. \qquad (2)$$

with

$$\beta(t) = \frac{C+mF}{r+m}e^{-mt} - \frac{\theta C}{r} \text{ and } \gamma(t) = \alpha + (1-\alpha)e^{-mt}.$$

Leland's Approximation.

▷ Restrict bankruptcy policies to hitting times

$$\tau_B = \inf\{t : V_t \le V_B\}$$

and consider

$$\begin{cases} \max_{\tau_B} \left(E(V, \tau_B) = v(V, \tau_B) - D(V, \tau_B) \right), \\ v(V, \tau_B) - D(V, \tau_B) \ge 0. \end{cases}$$

▷ In a large class of models, we have an explicit computation of $E(V, \tau_B)$,

$$V + \frac{\theta C}{r} - \frac{C + mF}{r + m} + \mathbb{E}\left[e^{-r\tau_B} (\beta(\tau_B) - \gamma(\tau_B) V^V_{\tau_B}) \right],$$

▷ *Smooth-fit principle*

- Solve

$$\frac{\partial E(., \tau_B)}{\partial V}(V_B, V_B) \ge 0.$$

- optimal threshold

$$V_B = \frac{y\frac{\theta C}{r} - y(m)\frac{C+mF}{r+m}}{1 - y(m) + \alpha(y(m) - y)}.$$

Solving the optimal problem.

▷ Introduce the process $(P_t)_{t\geq 0}$ defined by

$$P_t = \text{ess} \sup_{\tau \in \mathcal{T}_{t,\infty}} \mathbb{E}\left[e^{-r\tau}(\beta(\tau) - \gamma(\tau)V_\tau)|\mathcal{F}_t\right]. \quad (3)$$

▷ The process $(P_t)_{t\geq 0}$ is the Snell envelope of the process $(e^{-rt}(\beta(t) - \gamma(t)V_t))_{t\geq 0}$, that is the smallest supermartingale that dominates $(e^{-rt}(\beta(t) - \gamma(t)V_t)_{t\geq 0}$.

▷ The strong Markov property gives the relation

$$P_t = e^{-rt}P(t, V_t)$$

where

$$P(t, V) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau}(\beta(t+\tau) - \gamma(t+\tau)V_\tau^V)\right]. \quad (4)$$

▷ The option value P defined in (2) satisfies

$$P(V) = P(0, V)$$

deserves particular comments. On the one hand, if m is sufficiently large the put option (2) associated to the irreversible bankruptcy decision has a negative payoff and must be never exercised (▷ Stopping region

$$S = \{(t, V) \in [0, \infty) \times (0, \infty), P(t, V) = \beta(t) - \gamma(t)V\}$$

▷ Optimal stopping time

$$\tau_S(t, V) = \inf\{s \geq 0, (t+s, V_s^V) \in S\}.$$

Solving the optimal problem (2).

▷ The t-sections S_t of the stopping region S.

$$S_t = \{V > 0 \,|\, P(t,V) = \beta(t) - \gamma(t)V\}.$$

▷ Optimal bankruptcy will occur for $V \in S_0$.

▷ Analytical characterization.
$P(t,.)$ is differentiable and P satisfies almost everywhere

$$\max(\frac{\partial P}{\partial t} + AP - rP, (\beta(.) - \gamma(.).) - P) = 0$$

▷ Shape of the t-sections.
Let $\bar{t} = \frac{1}{m} \ln \left(\frac{r(C + mF)}{\theta C(r+m)} \right)$ then,

(i) For $t \geq \bar{t}$, the t-sections S_t are empty.

(ii) For all $0 \leq t < \bar{t}$, the t-sections are left connected, that is there exists a strictly positive function $b^*(t)$ such that $S_t =]0, b^*(t)]$.

▷ As a consequence, $E(V) > E(V, \tau_B)$. The Leland approximation underestimates the equity value, to what extent?

Short term maturity

▷ When $m = 0$, the average maturity is infinite and the optimal stopping problem (2) coincides with the Leland solution.

▷ When $m = \infty$, (the average maturity is zero) the value of the put option (2) tends to zero. Therefore, the equity value E satisfies

$$\lim_{m \to \infty} E = V + \frac{\theta C}{r} - F.$$

▷ When $m = \infty$, the Leland equity value is

$$\begin{aligned} E(V, \tau_B) &= V + \frac{\theta C}{r} - F + \mathbb{E}\left[e^{-r\tau_B}(\beta(\tau_B) - \gamma(\tau_B)V^V_{\tau_B})\right] \\ &= V + \frac{\theta C}{r} - F - \left(\alpha \frac{F}{1-\alpha} + \frac{\theta C}{r}\right)\left(\frac{V(1-\alpha)}{F}\right)^y \end{aligned}$$

▷ When $m = \infty$, the optimal bankruptcy threshold is thus $F - \frac{\theta C}{r}$.

▷ When $m = \infty$, the Leland approximation threshold is $F/(1-\alpha)$.

Numerical Results

$\sigma = 0.2$, $r = 7.5\%$, $\delta = 7\%$, $\alpha = 50\%$, $\theta = 35\%$
$V = 100$, $F = 40$ and $C = 5$

m	∞	4	2	1
$E(V, V_B)$	39.05	59.20	63.30	66.83
$E(V)$	83.33	82.84	82.37	81.47
V_B	79.9	60.08	54.40	48.15
V^*	16.67	17.13	17.56	18.38

m	0.2	0.1	0.05
$E(V, V_B)$	69.36	67.57	65
$E(V)$	76.06	71.90	67.34
V_B	34.87	31.20	29
V^*	22.65	25.1	26.3