

# Hedging Credit Derivatives in Intensity Based Models

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## Overview

- We will be focussed on **intensity** based approaches for the joint pricing of equity and credit derivatives.
- There are three parts to this talk:
  1. **Replicating** the payoff to a defaultable bond by dynamic trading in an equity option and its underlying.
  2. Replicating the payoff to a digital default claim by dynamic trading in an option and its underlying.
  3. Semi-robust replication of a digital default claim using a static position in a single name variance swap, options, and dynamic trading in their underlying.

## Part I: Replicating a Defaultable Bond

- Inspired by a prior conjecture by Samuelson (1973), Merton (1976) valued a call option in closed form in an extension of the standard Black Scholes model which allows the stock price to jump to zero at an independent exponential time.
- Define a **defaultable bond** as a claim that pays one dollar at  $T$  if no default occurs prior to  $T$  and which pays zero otherwise.
- We will show how to replicate the payoff to this defaultable bond in the Black Scholes model with jump-to-default.
- The replicating strategy involves dynamic trading in just stocks and **calls**; no position in a riskfree asset is required.

## Assumptions

- Consider a fixed time interval  $[0, T]$  and assume that the **riskfree rate is constant** at some finite  $r \in \mathbb{R}$  over this period.

- Let:

$$B_t \equiv e^{-r(T-t)}$$

denote the price at time  $t \in [0, T]$  of a bond paying one dollar at  $T$  with certainty.

- Also trading is a stock which for simplicity pays **no dividends** over  $[0, T]$ .

## Assumptions (Con'd)

- Fix a probability space and let  $\mathbb{P}$  denote statistical probability measure.
- Let  $S_t$  denote the spot price of the stock with  $S_0$  a known positive constant.
- Under  $\mathbb{P}$ , let  $S$  solve the following stochastic differential equation:

$$dS_t = \mu_t S_{t-} dt + \sigma S_{t-} dW_t - S_{t-} dN_t, \quad t \in [0, T],$$

where  $\sigma > 0$  is the positive constant volatility of the stock.

- The subscript  $t-$  on  $S$  indicates the pre-jump stock price at  $t$ .
- The drift process  $\mu_t$  is restricted so that  $S$  can neither explode nor hit zero by diffusion.
- The (doubly stochastic) Poisson process  $N$  has statistical arrival rate  $\alpha_t \geq 0$ .
- When  $N$  jumps from 0 to 1, the stock price  $S$  drops to zero and stays there.

## Black Scholes Pricing Model

- First consider the famous case where the default arrival process  $\alpha$  is identically zero.
- It is very well known that for bounded  $\mu$ , the arbitrage-free value of a European call at time  $t \in [0, T]$  is given by the Black Scholes call formula:

$$C_t = N(d_1(S_t, B_t))S_t - KN(d_2(S_t, B_t))B_t,$$

where the functions  $d_1$  and  $d_2$  are defined by:

$$d_1(S, B) \equiv \frac{\ln\left(\frac{S}{BK}\right) + \frac{\sigma^2}{2}(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2(S, B) \equiv \frac{\ln\left(\frac{S}{BK}\right) - \frac{\sigma^2}{2}(T - t)}{\sigma\sqrt{T - t}}.$$

## BS/Merton Replicating Argument

- Recall the Black Scholes call formula:

$$C_t = N(d_1(S_t, B_t))S_t - KN(d_2(S_t, B_t))B_t, \quad t \in [0, T].$$

- This formula tells us that the payoff from a static position in one call option can be replicated by:
  1. Holding  $N(d_1(S_t, B_t))$  shares at each time  $t \in [0, T]$
  2. Shorting  $KN(d_2(S_t, B_t))$  bonds at each time  $t \in [0, T]$ .
- Since the positions in each of the two assets depends on both asset prices and time, the trading strategy is **dynamic**.
- It is also well known that this dynamic trading strategy is **self-financing**.

## BS/Merton Hedging

- As originally suggested by Merton, a static position in one call option combined with a short position in  $N(d_1(S_t, B_t))$  shares is locally **riskless** and therefore must earn the riskfree rate  $r$  (under  $\mathbb{P}$ !).
- If we scale the call and share positions by the same factor, then the position is still locally riskless.
- If the scale factor varies stochastically over time in a non-anticipating way, the position in calls and shares is **still** locally riskless.

## BS/Merton Bond Replication

- Recall that the BS call formula is  $C_t = N(d_1(S_t, B_t))S_t - KN(d_2(S_t, B_t))B_t$ .
- Suppose that we “solve” the BS call formula for the bond price  $B$ :

$$B_t = \frac{C_t}{KN(d_2(S_t, B_t))} - \frac{N(d_1(S_t, B_t))}{KN(d_2(S_t, B_t))}S_t.$$

- This formula tells us that the payoff from a static position in one bond can be replicated by:
  1. Holding  $\frac{1}{KN(d_2(S_t, B_t))}$  calls at each time  $t \in [0, T]$
  2. Shorting  $\frac{N(d_1(S_t, B_t))}{KN(d_2(S_t, B_t))}$  shares at each time  $t \in [0, T]$ .
- Since the positions in call and stock depends on time and the stock and bond prices, the trading strategy is dynamic.
- This strategy is also self-financing.
- The general result is that a static position in any one of the 3 assets can be replicated by a self-financing dynamic trading strategy in the other two.

## BS/Merton Bond Replication

- In many real world problems, the prices of both the target and basis assets are readily observed (say on Bloomberg).
- The main objectives are either:
  1. to hedge i.e. eliminate variance, or
  2. to make model-based forecasts on what one price will be for a given move in the other asset prices.
- With all of these observations in mind, let's see how these results change if a jump to default is added to Black Scholes.

## Black Scholes Model with Jump to Default

- Recall that we assumed that under the statistical measure  $\mathbb{P}$ , the stock price  $S$  follows:

$$dS_t = \mu_t S_{t-} dt + \sigma S_{t-} dW_t - S_{t-} dN_t, \quad t \in [0, T].$$

- We now suppose that the real-world default arrival rate process  $\alpha$  is **strictly positive** at all times.
- Let  $\tau_1$  denote the random jump time of  $S$  to default.
- Suppose that there exists a defaultable bond:
  - which pays \$1 at  $T$  if  $\tau_1 > T$
  - and which pays \$ 0 otherwise.

## Defaultable Bond Price Dynamics

- Suppose for now that the defaultable bond price can be directly observed (on Bloomberg of course).
- Suppose that in the past (i.e. before time 0), the defaultable bond price has enjoyed **constant** exponential growth rate  $r + \lambda$ .
- Based on these observations, we boldly predict that:

$$D_t \equiv e^{-(r+\lambda)(T-t)} \mathbf{1}(\tau_1 > t), \quad t \in [0, T],$$

where recall that  $\tau_1$  is the random jump time of the stock price to zero.

- Since the only jump allowed in the stock price is the one to zero,  $\tau_1$  is the first and only jump time of both the stock price process and the defaultable bond price process. Both processes jump to zero at time  $\tau_1$ , which can occur before, at, or after  $T$ .

## Merton 1976 Call Pricing

- Merton (1976) valued a call when the Black Scholes model is extended by allowing the stock price to jump to zero at an independent exponential time.
- The revised pricing formula is  $C_t = N(d_1(S_t, D_t))S_t - KN(d_2(S_t, D_t))D_t$ , where recall that the functions  $d_1$  and  $d_2$  are:

$$d_1(S, D) \equiv \frac{\ln\left(\frac{S}{DK}\right) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2(S, D) \equiv \frac{\ln\left(\frac{S}{DK}\right) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}.$$

- In words, the defaultable bond price  $D_t \equiv e^{-(r+\lambda)(T-t)} \mathbf{1}(\tau_1 > t)$  replaces the default-free bond price  $B_t \equiv e^{-r(T-t)}$  everywhere.
- Since  $S$  and  $D$  can now both be zero, we now need to know the call value at the point  $(S, D) = (0, 0)$  where the functions  $d_1$  and  $d_2$  are not defined.
- The obvious solution is to use the above pricing formula only for  $t \in [0, \tau_1 \wedge T)$ .
- For times  $t$  in the possibly empty set  $[\tau_1, T]$ , we directly define  $C_t \equiv 0$ .

## Merton 1976 Call Replication

- Recall Merton's extended pricing formula for the European call:

$$C_t = \begin{cases} N(d_1(S_t, D_t))S_t - KN(d_2(S_t, D_t))D_t, & \text{if } t \in [0, \tau_1 \wedge T) \\ 0 & \text{if } t \in [\tau_1, T]. \end{cases}$$

- When considered as a **replication recipe**, this formula clearly states that the payoff from a static position in one call option can be replicated by:
  1. Holding  $N(d_1(S_t, D_t))$  (defaultable) shares at each time  $t \in [0, \tau_1 \wedge T)$ .
  2. Shorting  $KN(d_2(S_t, D_t))$  defaultable bonds at each time  $t \in [0, \tau_1 \wedge T)$ .
  3. Holding no shares or defaultable bonds for  $t \in [\tau_1, T]$ .
- This dynamic strategy can be proven to be both self-financing and replicating. Intuitively, if no default occurs over  $[0, T]$ , then the Black Scholes model reigns with the riskfree rate given by  $r + \lambda$ . Hence, standard results can be used to show that the strategy is self-financing and replicating in this case. If a default does occur before  $T$ , then the defaultable stock-bond position jumps to zero at the default time, thereby replicating the call value.

## “Efficient” Call Replication

- It is surprising that the call payoff can be perfectly replicated by dynamic trading in just **2** assets when both diffusion and jump components are present.
- Standard arbitrage pricing theory implies that when the Black Scholes model is extended to include a jump to default, **3** assets are needed to span the payoff of a given equity derivative. Indeed, one cannot replicate the payoff to a European **put** option using just dynamic trading in defaultable stocks and bonds.
- What is special about calls is that  $C_t = 0$  when  $(S_t, D_t) = (0, 0)$  and that  $C_t > 0$  when  $(S_t, D_t) > (0, 0)$ , except possibly at  $t = T$ .
- Other equity derivatives have this property, eg. claims paying  $S_T^p$  for  $p \geq 0$ .
- For all such claims, if the stock position is used to neutralize the Brownian exposure of the portfolio, and the defaultable bond is used to finance the required changes in this stock position, then essentially by luck, the strategy also neutralizes the Poisson component.

## Replicating Defaultable Bonds

- Suppose that we “solve” Merton’s call formula for the defaultable bond price  $D$ :

$$D_t = \frac{C_t}{KN(d_2(S_t, D_t))} - \frac{N(d_1(S_t, D_t))}{KN(d_2(S_t, D_t))} S_t.$$

- This formula tells us that the payoff from a static position in one defaultable bond can be replicated by:
  1. Holding  $\frac{1}{KN(d_2(S_t, D_t))}$  calls at each time  $t \in [0, T]$ .
  2. Shorting  $\frac{N(d_1(S_t, D_t))}{KN(d_2(S_t, D_t))}$  shares at each time  $t \in [0, T]$ .
- This trading strategy in calls and stocks is **dynamic, self-financing, and replicating**.
- If we didn’t know the defaultable bond price ex-ante, it wouldn’t matter, since in a world of no arbitrage, we could numerically imply it out from  $S$  and  $C$ .

## Summary

- An extension of the Black Scholes model was considered in which the stock price can jump to zero at an independent exponential time.
- In this setting, the payoff to a static position in one defaultable bond can be replicated by a dynamic self-financing trading strategy in just calls and stocks.
- It is well known that the replication of a contingent claim written on the path of a single Brownian motion requires dynamic trading in one risky asset and one riskless asset.
- When one adds a jump to default to a Brownian motion, then spanning of the payoff from an arbitrary contingent claim generally requires dynamic trading in *two* risky assets and one riskless asset.
- The triple consisting of the call, the stock, and the defaultable bond are different. As they are all worth zero in the event of a default, any one can be replicated by dynamic trading in the other two. No position in a riskless asset is required.

## Future Research

- Future research can be directed towards generalizing this result by allowing more complicated dynamics for the asset prices.
- By exploiting both the linear homogeneity of the call payoff and the proportional dynamics of the defaultable stock and bond prices, the same result holds if  $r$  and  $\lambda$  are each allowed to follow (possibly correlated) Gaussian processes.
- It is not clear whether the parsimony of the hedge survives a restriction to non-negative and non-deterministic processes for  $r$  and  $\lambda$ .
- In the next part, we show that the same qualitative result holds when instantaneous volatility is a known function of stock price and time.

## Part II: Replicating a Digital Default Claim

- By definition, a digital default claim pays one dollar at the default time if a default occurs and zero otherwise.
- To value this claim by no arbitrage, we assume that the stock price is a continuous process up to the default time. At the default time, the stock price again jumps to zero.
- Assuming that the instantaneous volatility is a known function of the stock price and time, we show how to replicate the payoff to a digital default claim by dynamic trading in a European call option and its underlying stock.

# Assumptions

- We assume frictionless markets in a European call option, its underlying stock, and a riskless asset. The call has strike price  $K$  and maturity  $T$ .
- For simplicity, we assume that over the option's life, we have a constant riskfree interest rate  $r$ , a constant dividend yield  $q$  (paid continuously over time), and a continuous stock price process prior to default.
- When default occurs, the stock price  $S$  drops to zero. Hence, under the statistical probability measure  $\mathbb{P}$ , we have the following dynamics:

$$dS_t = \mu_t S_{t-} dt + \sigma(S_{t-}, t) S_{t-} dW_t - S_{t-} dN_t, \quad t \in [0, T],$$

where  $W$  is a  $\mathbb{P}$  standard Brownian motion and  $N$  is a doubly stochastic Poisson process with statistical arrival rate  $\alpha_t$ .

- The processes  $\mu$  and  $\alpha$  need not be known, but the volatility function  $\sigma(S, t) : \mathbb{R}^+ \times [0, T] \mapsto \mathbb{R}^+$  must be known.

## Default Indicator

- Recall the SDE:

$$dS_t = \alpha_t S_{t-} dt + \sigma(S_{t-}, t) S_{t-} dW_t - S_{t-} dN_t, \quad t \in [0, T].$$

- Let  $D_t \equiv 1(N_t \geq 1)$  be the default indicator process. If  $N$  jumps from 0 to 1 at some time  $t$ , then the SDE indicates that the stock price drops from  $S_{t-}$  to zero and stays there afterwards.
- As a result, we also have:

$$dS_t = \alpha_t S_{t-} dt + \sigma(S_{t-}, t) S_{t-} dW_t - S_{t-} dD_t, \quad t \in [0, T]. \quad (1)$$

## Delta Hedging a Long Call

- What is the P&L if an investor buys a European call and delta hedges it?
- Let  $V(S, t) : \mathbb{R}^+ \times [0, T] \mapsto \mathbb{R}^+$  be a  $C^{2,1}$  function. Applying Itô to  $V_t \equiv V(S_t, t)e^{r(T-t)}$ :

$$\begin{aligned}
 V(S_T, T) &= V(S_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_{t-}, t) dS_t \\
 &+ \int_0^T e^{r(T-t)} \left[ \frac{\sigma^2(S_{t-}, t) S_{t-}^2}{2} \frac{\partial^2 V}{\partial S^2}(S_{t-}, t) - rV(S_{t-}, t) + \frac{\partial V}{\partial t}(S_{t-}, t) \right] dt \\
 &+ \int_0^T e^{r(T-t)} \left[ V(0, t) - V(S_{t-}, t) - \frac{\partial}{\partial S} V(S_{t-}, t)(0 - S_{t-}) \right] dD_t.
 \end{aligned}$$

- Subtracting and adding the stock carrying cost implies:

$$\begin{aligned}
 V(S_T, T) &= V(S_0, 0)e^{r(T-t)} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_{t-}, t) [dS_t - (r - q)S_{t-}dt] \\
 &+ \int_0^T e^{r(T-t)} \left[ \frac{\sigma^2(S_{t-}, t) S_{t-}^2}{2} \frac{\partial^2 V}{\partial S^2}(S_{t-}, t) + (r - q)S_{t-} \frac{\partial V}{\partial S}(S_{t-}, t) - rV(S_{t-}, t) + \frac{\partial V}{\partial t}(S_{t-}, t) \right] dt \\
 &+ \int_0^T e^{r(T-t)} \left[ V(0, t) - V(S_{t-}, t) + \frac{\partial}{\partial S} V(S_{t-}, t) S_{t-} \right] dD_t.
 \end{aligned}$$

## Martingale Representation

- Recall the representation:

$$\begin{aligned}
 V(S_T, T) &= V(S_0, 0)e^{r(T-t)} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_{t-}, t) [dS_t - (r - q)S_{t-}dt] \\
 &+ \int_0^T e^{r(T-t)} \left[ \frac{\sigma^2(S_{t-}, t)S_{t-}^2}{2} \frac{\partial^2 V}{\partial S^2}(S_{t-}, t) + (r - q)S_{t-} \frac{\partial V}{\partial S}(S_{t-}, t) - rV(S_{t-}, t) + \frac{\partial V}{\partial t}(S_{t-}, t) \right] dt \\
 &+ \int_0^T e^{r(T-t)} \left[ V(0, t) - V(S_{t-}, t) + \frac{\partial}{\partial S} V(S_{t-}, t) S_{t-} \right] dD_t.
 \end{aligned}$$

- Now, by choosing  $V(S, t)$  to solve the parabolic P.D.E.:

$$\frac{\sigma^2(S, t)S^2}{2} \frac{\partial^2 V}{\partial S^2}(S, t) + (r - q)S \frac{\partial V}{\partial S}(S, t) - rV(S, t) + \frac{\partial V}{\partial t}(S, t) = 0, \text{ and } V(S, T) = (S - K)^+,$$

- we get:

$$\begin{aligned}
 (S_T - K)^+ &= V(S_0, 0)e^{rT} + \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_{t-}, t) [dS_t - (r - q)S_{t-}dt] \\
 &+ \int_0^T e^{r(T-t)} \left[ \frac{\partial}{\partial S} V(S_{t-}, t) S_{t-} - V(S_{t-}, t) \right] dD_t.
 \end{aligned}$$

## P&L from Delta Hedging a Long Call

- Consider buying a call at  $t = 0$  for  $C_0$  and shorting  $\frac{\partial V}{\partial S}(S_{t-}, t)$  shares  $\forall t \in [0, T]$ .
- The terminal P&L arising from this strategy is:

$$P\&L_T = -C_0 e^{rT} + (S_T - K)^+ - \int_0^T e^{r(T-t)} \frac{\partial V}{\partial S}(S_{t-}, t) [dS_t - (r - q)S_{t-} dt].$$

- From the last equation on the last page, an alternative representation is available:

$$P\&L_T = -C_0 e^{rT} + V(S_0, 0) e^{rT} + \int_0^T e^{r(T-t)} \left[ \frac{\partial}{\partial S} V(S_{t-}, t) S_{t-} - V(S_{t-}, t) \right] dD_t.$$

- $\frac{\partial}{\partial S} V(S_{t-}, t) S_{t-} - V(S_{t-}, t)$  is recognized as the positive dollar amount kept in the riskless asset at time  $t \in [0, T)$  when delta hedging a long call under no jumps.
- If the stock price never jumps, then  $dD_t = 0$  for all  $t \in [0, T]$  and the last term in the 2nd equation vanishes. In contrast, if the stock price drops to zero at some time  $t$  prior to expiry, then the value of the long call and its delta hedge both vanish. The investor is left holding a positive position in a riskless asset which can be liquidated if desired.

## Implications of Delta Hedging a Long Call

- The analysis on the previous slide implies that the realized P&L increases by  $\frac{\partial}{\partial S}V(S_{t-},t)S_{t-} - V(S_{t-},t)$  at the default time  $t$ .

- Recall our last expression for the profit and loss:

$$P\&L_T = -C_0e^{rT} + V(S_0,0)e^{rT} + \int_0^T e^{r(T-t)} \left[ \frac{\partial}{\partial S}V(S_{t-},t)S_{t-} - V(S_{t-},t) \right] dD_t.$$

- Since the last term is nonnegative, no arbitrage requires that  $C_0 > V(S_0,0)$ .
- Adding a jump to 0 unambiguously raises the value of a call from  $V(S_0,0)$  to  $C_0$ .
- Clearly,  $C_0 - V(S_0,0)$  is the positive price one must pay initially to get a claim paying zero if no default occurs and paying out  $\frac{\partial}{\partial S}V(S_{t-},t)S_{t-} - V(S_{t-},t)$  dollars at the default time  $t$  if this occurs before  $T$ .
- No arbitrage further requires that there must be some time  $t \in [0, T]$  and some stock price  $S$  such that  $\frac{\partial}{\partial S}V(S,t)S - V(S,t) > [C_0 - V(S_0,0)]e^{rt}$ . Otherwise, an arbitrage profit can be made by shorting the call and delta hedging it as if no jumps can occur.

## Dynamic Replication of a Digital Default Claim

- We have shown that delta hedging a long call requires paying  $\$C_0 - V(S_0, 0)$  initially in return for  $\$ \frac{\partial}{\partial S} V(S_{t-}, t) S_{t-} - V(S_{t-}, t)$  received iff default occurs by  $T$ .
- By definition, a digital default claim requires an up front premium payment and pays out one dollar at the default time if and only if a default occurs before  $T$ .
- To find the fair initial premium, let's rescale the strategy by  $\frac{1}{\frac{\partial}{\partial S} V(S_{t-}, t) S_{t-} - V(S_{t-}, t)}$  at each  $t$  prior to default. Thus, prior to default, the strategy consists of:
  - long  $\frac{1}{\frac{\partial}{\partial S} V(S_{t-}, t) S_{t-} - V(S_{t-}, t)}$  calls of maturity  $T$
  - short  $\frac{\frac{\partial V}{\partial S}(S_{t-}, t)}{\frac{\partial}{\partial S} V(S_{t-}, t) S_{t-} - V(S_{t-}, t)}$  shares with the riskless asset financing all changes in positions.
- Clearly, the upfront initial premium is also obtained by scaling. We conclude that the initial arbitrage-free price of the digital default claim is given by:

$$\frac{C_0 - V(S_0, 0)}{\frac{\partial}{\partial S} V(S_0, 0) S_0 - V(S_0, 0)}.$$

## Summary

- We showed that the payoff to a digital default claim can be replicated by a dynamic trading strategy in calls, shares, and the riskfree asset.
- In contrast to Part I of my talk (and all research that I am aware of), we did not have to specify the risk-neutral arrival rate of default.
- In common with Part I of my talk, we did have to know the volatility process. We relax this assumption in the next part of my talk.

## Future Research

- Future research can be directed towards generalizing this result by allowing more complicated dynamics for the asset prices.
- The same qualitative results hold under deterministic  $r$  and  $q$ , even if the stock volatility depends on the paths of traded asset prices.
- As more uncertainty is added, eg. an independent source of variation in volatility, then more assets are needed to hedge.
- It would be interesting to consider newer derivatives such as equity default swaps as hedge instruments. We will consider variance swaps as hedge instruments in the next part.

## Part III: Default and Variance Swaps

- Define a digital default claim as a security that pays \$1 at a fixed time  $T$  if default occurs before  $T$  and which pays \$0 otherwise.
- Under the assumptions of Part 1, the payoff to a digital default claim can be replicated by buying a default-free bond and synthetically shorting a defaultable bond.
- In this part, we will show that under different assumptions than Part 1, the payoff to a digital default claim can alternatively be replicated by combining static positions in variance swaps and European options with dynamic trading in the underlying stock.

## Assumptions for Part III

- We assume that investors can take a static position in  $T$  maturity zero coupon bonds and  $T$  maturity European stock options of all strikes  $K > 0$ .
- Letting  $S_T$  denote the terminal stock price, any twice differentiable payoff  $f(S_T) = f(\kappa) + f'(\kappa)[(S_T - \kappa)^+ - (\kappa - S_T)^+] + \int_0^\kappa f''(K)(K - S_T)^+ dK + \int_\kappa^\infty f''(K)(S_T - K)^+ dK$ , where the put call separator  $\kappa \geq 0$  is arbitrary.
- It follows that the payoff  $f(S_T)$  can be statically replicated at  $T$  by initially buying:
  - $f(\kappa)$  zero coupon bonds,
  - $f'(\kappa)$  calls of strike  $\kappa$ ,
  - $-f'(\kappa)$  puts of strike  $\kappa$ ,
  - $f''(K)dK$  puts of all strikes  $K < \kappa$ , and
  - $f''(K)dK$  calls of all strikes  $K > \kappa$ .

## Stock Price Dynamics for Part III

- We furthermore assume that under the statistical probability measure  $\mathbb{P}$ , the stock price  $S$  is a positive continuous process with unknown stochastic drift and volatility before and after the default time.
- When a default occurs, the stock price  $S$  drops by a fixed known percentage of its pre-default value.
- We do not allow the stock price to drop to zero or below.
- Notice that the assumptions concerning default differ from those made in Parts I and II.

## Formal Stock Price Dynamics

- Let  $N$  be a doubly stochastic Poisson process with compensating process  $\alpha$ . A default occurs when  $N$  jumps from 0 to 1.
- Let  $D_t \equiv 1(N_t \geq 1)$  be the default indicator process and again let  $\tau_1$  denote the random default time, possibly infinite.
- Under the statistical probability measure  $\mathbb{P}$ , the stock price  $S$  solves:

$$dS_t = \mu_t S_{t-} dt + \sigma_t S_{t-} dW_t + S_{t-} (e^j - 1) dD_t, \quad t \in [0, T],$$

where  $S_0 > 0$  is known and  $W$  is a  $\mathbb{P}$  standard Brownian motion.

- We do not require knowledge of the stochastic processes  $\mu$ ,  $\sigma$ , and  $\alpha$ .
- If  $D$  jumps from 0 to 1 at time  $\tau$ , then the stock price drops from  $S_{\tau-}$  to  $S_{\tau-} e^j$ , where  $j \neq 0$  is a known finite constant. We have in mind that  $j < 0$ , but the analysis goes through if  $j > 0$ .
- After the default time  $\tau$ , the stock price process  $S$  is continuous over time, possibly constant.

## More Assumptions

- By Itô's formula for semi-martingales, the log price dynamics are given by:

$$d \ln S_t = \left( \mu_t - \frac{\sigma_t^2}{2} - (e^j - 1 - j)\alpha_t 1(t \leq \tau_1) \right) dt + \sigma_t dW_t + j dD_t, \quad t \in [0, T].$$

- Assume that investors can dynamically trade the stock of the company without frictions.
- We also assume zero interest rates and stock dividends over  $[0, T]$  for simplicity.

## Variance Swap

- We furthermore assume that investors can take a static position in a continuously monitored variance swap of maturity  $T$ . For each dollar of notional, the floating part of the time  $T$  payoff of a variance swap is the uncapped amount:

$$s_T^2 \equiv \frac{1}{T} \int_0^T (d \ln S_t)^2 = \frac{1}{T} \left[ \int_0^T \sigma_t^2 dt + D_T j^2 \right].$$

- For each dollar of notional, the variance swap payoff is determined by subtracting the fixed part of the payoff  $s_0^2$  from  $s_T^2$ , where the initial variance swap rate  $s_0$  is chosen so that the variance swap has zero cost to enter.
- When the stock price follows a positive continuous process, the payoff to a variance swap can be perfectly replicated by combining a static position in European options of all strikes with dynamic trading in the underlying stock. Our stock price process is positive but it is not continuous and hence the variance swap is not a redundant asset in our setting.

## Replicating a Digital Default Claim

- We want to replicate the payoff  $D_T$  where recall that  $\{D_t \equiv 1(N_t \geq 1), t \in [0, T]\}$  is the default indicator process.
- Let  $f(x) \equiv \ln(S_0/x), x > 0$ . Since  $f$  is  $C^2$ , applying Itô's formula to  $f(S_t)$  yields:

$$\ln\left(\frac{S_0}{S_T}\right) = \int_0^T \frac{-1}{S_{t-}} dS_t + \frac{1}{2} \int_0^T \frac{1}{S_{t-}^2} d\langle S \rangle_t^c + \int_0^T \left[ \ln\left(\frac{S_0}{S_{t-}e^j}\right) - \ln\left(\frac{S_0}{S_{t-}}\right) + \frac{1}{S_{t-}} S_{t-}(e^j - 1) \right] dD_t.$$

- Simplifying and re-arranging:

$$\ln\left(\frac{S_0}{S_T}\right) + \int_0^T \frac{1}{S_{t-}} dS_t - \frac{1}{2} \int_0^T \sigma_t^2 dt = \int_0^T [e^j - 1 - j] dD_t = [e^j - 1 - j] D_T,$$

since  $D_0 = 0$ .

## Replicating a Digital Default Claim (Con'd)

- Recall that we have:

$$\ln\left(\frac{S_0}{S_T}\right) + \int_0^T \frac{1}{S_{t-}} dS_t - \frac{1}{2} \int_0^T \sigma_t^2 dt = [e^j - 1 - j] D_T.$$

- Suppose that we subtract  $\frac{1}{2}D_T j^2$  from both sides:

$$\ln\left(\frac{S_0}{S_T}\right) + \int_0^T \frac{1}{S_{t-}} dS_t - \frac{1}{2} \int_0^T \sigma_t^2 dt - \frac{1}{2} D_T j^2 = \left[ e^j - 1 - j - \frac{j^2}{2} \right] D_T.$$

Dividing by  $e^j - 1 - j - \frac{j^2}{2}$  and substituting in the floating part of the variance swap payoff implies:

$$D_T = \alpha \ln\left(\frac{S_0}{S_T}\right) + \int_0^T \frac{\alpha}{S_{t-}} dS_t - \frac{\alpha}{2} T s_T^2, \text{ where } \alpha \equiv \frac{1}{e^j - 1 - j - \frac{j^2}{2}}.$$

## Replicating a Digital Default Claim (Con'd)

- Recall our current representation of the digital default claim payoff:

$$D_T = \alpha \ln \left( \frac{S_0}{S_T} \right) + \int_0^T \frac{\alpha}{S_{t-}} dS_t - \frac{\alpha}{2} T s_T^2, \text{ where } \alpha \equiv \frac{1}{e^j - 1 - j - \frac{j^2}{2}}.$$

- The log payoff can be replicated by a static position in options:

$$\alpha \ln \left( \frac{S_0}{S_T} \right) = -\frac{\alpha}{S_0} [(S_T - S_0)^+ - (S_0 - S_T)^+] + \int_0^{S_0} \frac{\alpha}{K^2} (K - S_T)^+ dK + \int_{S_0}^{\infty} \frac{\alpha}{K^2} (S_T - K)^+ dK.$$

- Substitution leads to our final representation of the digital default claim payoff:

$$D_T = -\frac{\alpha}{S_0} [(S_T - S_0)^+ - (S_0 - S_T)^+] + \int_0^{S_0} \frac{\alpha}{K^2} (K - S_T)^+ dK + \int_{S_0}^{\infty} \frac{\alpha}{K^2} (S_T - K)^+ dK \\ + \int_0^T \frac{\alpha}{S_{t-}} dS_t - \frac{\alpha}{2} T s_T^2.$$

## Replicating the Digital Default Claim

- Recall our final representation of the digital default claim payoff:

$$D_T = -\frac{\alpha}{S_0}[(S_T - S_0)^+ - (S_0 - S_T)^+] + \int_0^{S_0} \frac{\alpha}{K^2}(K - S_T)^+ dK + \int_{S_0}^{\infty} \frac{\alpha}{K^2}(S_T - K)^+ dK \\ + \int_0^T \frac{\alpha}{S_{t-}} dS_t - \frac{\alpha}{2} T s_T^2.$$

- Thus, the payoff on the digital default claim can be replicated by:
  1. a static position in  $-\frac{\alpha}{S_0}$  ATM calls
  2. a static position in  $\frac{\alpha}{S_0}$  ATM puts
  3. a static position in  $\frac{\alpha}{K^2} dK$  puts for all strikes  $K < S_0$
  4. a static position in  $\frac{\alpha}{K^2} dK$  calls for all strikes  $K > S_0$
  5. a dynamic position holding  $\frac{\alpha}{S_{t-}}$  shares at each  $t \in [0, T]$
  6. a static position in variance swaps with a notional of  $-\frac{\alpha}{2} T$ .

## Pricing the Digital Default Claim

- Again recall our final representation of the digital default claim payoff:

$$D_T = -\frac{\alpha}{S_0}[(S_T - S_0)^+ - (S_0 - S_T)^+] + \int_0^{S_0} \frac{\alpha}{K^2}(K - S_T)^+ dK + \int_{S_0}^{\infty} \frac{\alpha}{K^2}(S_T - K)^+ dK + \int_0^T \frac{\alpha}{S_{t-}} dS_t - \frac{\alpha}{2} T s_T^2.$$

- Let  $\mathbb{Q}$  denote the risk-neutral measure (associated with the default-free bond as the numeraire.)
- Taking risk-neutral expected values on both sides, the risk-neutral probability of a default over  $[0, T]$  is given by:

$$\mathbb{Q}\{D_T = 1\} = -\frac{\alpha}{S_0}[C_0(S_0, T) - P_0(S_0, T)] + \int_0^{S_0} \frac{\alpha}{K^2} P_0(K, T) dK + \int_{S_0}^{\infty} \frac{\alpha}{K^2} C_0(K, T) dK - \frac{\alpha}{2} T s_0^2,$$

where  $C_0(K, T) \equiv E^{\mathbb{Q}}(S_T - K)^+$  is the initial price of a call of strike  $K$  and  $P_0(K, T) \equiv E^{\mathbb{Q}}(K - S_T)^+$  is the initial price of a put of strike  $K$ .

## Summary

- Assuming that the only possible jump in the log stock price has known size, we showed that the payoff to a digital default claim can be perfectly replicated by a static position in variance swaps and standard options combined with dynamic trading in the underlying asset.
- In contrast to any other work on credit derivatives replication, we made no assumptions on the real world or risk-neutral arrival rate process.
- In addition, we made no assumption on the real world or risk-neutral dynamics of instantaneous volatility, except that we required that stock prices always be positive.

## Future Research

- In general, one can also attempt to consider these results in the context of multiple underlying stocks.
- One can then attempt a unified framework for CDO's, single name CDS's, index CDS, single name options, index options, single name variance swaps, and index variance swaps (oh, and I forgot EDS!).