1 Exercises: multiparameter part II

1.1 Exercise

Suppose we observe independent $X_i \sim \text{Gamma}(1, \alpha_i), 1 \leq i \leq k$ (i.e. scale 1 but shape parameter $\alpha_i$). Set

1. Set

$$f(x_1, \ldots, x_k) = \left( x_1, \ldots, x_k, \sum_{i=1}^{k} x_i \right).$$

Show that the push-forward of the the original exponential family of distributions on $\mathbb{R}^k$ is an exponential family of distributions on $\mathbb{R}^{k+1}$. What is the sufficient statistic?

2. What is the dimension of the natural parameter space (i.e. how many parameters are there)?

3. What is the reference measure?

4. Suppose $g(x) = Ax + b$. Give a sufficient condition on $(A, b)$ so that the push forward of an exponential family is still an exponential family. Give an example of $(A, b)$ for which the push forward fails to be an exponential family.

1.2 Exercise

1. Give a general formula for

$$P_{\eta} \left( t_1(X) \in A \mid t_2(X) \right).$$

Show that it is an exponential family.

2. What is its sufficient statistic?

3. What is its reference measure? Be formal about it: what is the sample space? What is the measure?

1.3 Exercise

Finally, for the Ising model we see that the CGF of $x_i$ under this measure is

$$\log \left( e^{Q^1_1 + 2 \sum_{(i,k) \in E} Q^2_{ik} x_k} + e^{-Q^1_1 - 2 \sum_{(i,k) \in E} Q^2_{ik} x_k} \right) + C$$

Let’s write this CGF as $\Lambda \left( Q^1, Q^2 \mid x_{-i} \right)$.

This is the CGF of a $\{1, -1\}$ valued random variable with natural parameter

$$\eta(x_{-i}, Q^1, Q^2) = Q^1_i + 2 \sum_{(i,k) \in E} Q^2_{ik} x_k,$$

and counting measure on $\{-1, 1\}$ as reference measure.

The notation $\eta(x_{-i}, Q^1, Q^2)$ suggests that the natural parameter corresponding to sufficient statistic $x_i$, when conditioning on $x_{-i}$, has changed. Does this conflict with what we saw earlier about conditioning?
1.4 Exercise
Write the Ising model above as a Markov random field above. Be specific as possible.

1. What are the $f_A$’s?
2. What are the $\eta_A$’s?
3. What is the reference measure?

1.5 Exercise
1. Write out the pseudolikelihood for the Ising model as explicitly as possible.
2. Is it convex in $(Q^1, Q^2)$?
3. Describe a Newton-Raphson algorithm to estimate $(Q^1, Q^2)$ based on maximizing the pseudolikelihood. Be as specific as possible, i.e. compute gradients and Hessians as explicitly as possible.

1.6 Exercise
Consider an Ising model on $L$, the $100 \times 100$ lattice in $\mathbb{Z}^2$ with $Q^1 = \alpha \cdot 111^T, Q^2 = \beta \cdot 111^T$.

1. For $\beta = 0, \alpha = 1$, initialize the Gibbs sampler at some random initial condition. Run the Gibbs sampler Markov chain on $\{-1, 1\}^L$ for some time. What do you expect the binary image to look like?
2. Repeat for $\beta > 0$ and $\beta < 0$.
   Note: I am not asking for an exhaustive simulation, the goal is to just get the basic mechanics of a Gibbs sampler.