1 Exercises: one parameter part II

1.1 Exercise
Suppose $X \sim \text{Poisson}(\lambda)$, i.e. $\mathbb{E}[X] = \lambda$.

1. Compute $\Lambda^*(s)$ with $\Lambda$ the cumulant generating function of $X$ (i.e. consider the Poisson family to have carrier measure Poisson($\lambda$)).

2. For $\lambda = 4$, compute the Chernoff bound for $\mathbb{P}(X > 10)$.

1.2 Exercise
Suppose $X \sim \text{Exponential}(\mu)$, i.e. $\mathbb{E}[X] = \mu$.

1. Compute $\Lambda^*(s)$ with $\Lambda$ the cumulant generating function of $X$ (i.e. consider the carrier measure to be Exponential($\mu$)).

2. For $\mu = 4$, compute the Chernoff bound for $\mathbb{P}(X > 10)$.

1.3 Exercise
1. Show that
   $$\tilde{\Lambda}^n_\eta(t) = n\tilde{\Lambda}_\eta^*(t).$$

2. Show that
   $$n\tilde{\Lambda}_\eta^*(t) = \tilde{D}^n(t; \mathbb{E}_\eta(t(X)))/2.$$

1.4 Exercise
1. Consider the Gamma density with shape parameter $k$ and scale parameter $\theta$ and set $T \sim \text{Gamma}(\theta, k)$. The scale parameter $\theta$ is such that $\mathbb{E}(T) = \theta k$. Show that, for any $n$ we can write the distribution of $T$ as the distribution of the sample mean of $n$ IID samples from some other Gamma distribution. What are $\theta_n, k_n$?

2. Set $k = 5$ compute and plot the saddle point approximation for a scale of $\theta = 1$. That is, consider the family on $\mathbb{R}$
   $$\frac{d\mathbb{P}_\eta}{dx} = e^{-\eta x - \Lambda(\eta)} \cdot \frac{x^4}{\Gamma(5)} 1_{[0,\infty)}(x) \, dx$$
   with $k = 5$ and $\eta$ chosen so that $\theta(\eta) = 1$.

3. Repeat 2. for shape parameter $k = 12.3$.

4. Now fix the scale at 1 and consider the family
   $$\frac{d\mathbb{P}_\eta}{dx} = e^{\eta \log x - \Lambda(\eta)} e^{-x} 1_{[0,\infty)}(x) \, dx.$$
   What is the cumulant generating function of this family?
1.5 Exercise
1. Give an example of a one parameter exponential family that is subgaussian.
2. Give an example of a one parameter exponential family that is not subgaussian.

1.6 Exercise
As $\mu = \hat{\Lambda}(\eta)$, we can also work out the posterior density of $\mu$:

$$
\pi(\mu|y) = \pi(\eta(\mu)|y) \frac{d\eta(\mu)}{d\mu} = \frac{\pi(\eta(\mu)|y)}{\text{Var}_\eta(Y)}
$$

1. What is the natural parameter for the exponential family $\pi(\mu|y)$?
2. What is the reference measure for the exponential family $\pi(\mu|y)$?

1.7 Exercise
Suppose $Y \sim \text{Poisson}(\mu)$ and we place a Gamma prior on $\mu$ with density

$$
\pi(\mu) = \frac{1}{\Gamma(k)\theta^k} \mu^{k-1} e^{-\mu/\theta} 1_{[0,\infty)}(\mu) = \text{Gamma}(k,\theta).
$$

1. Show that

$$
\pi(\mu|y) = \text{Gamma} \left( y + k, \frac{\theta}{\theta + 1} \right).
$$

2. What is the posterior distribution of $\mu$ having observed $Y_1,\ldots,Y_n|\mu \overset{IID}{\sim} \text{Poisson}(\mu)$.

3. Suppose, instead, $Y_1,\ldots,Y_n \overset{IID}{\sim} \text{Poisson}(\mu_0)$ for some fixed $\mu_0$. Can you make sense of the density $\pi(\mu|y_1,\ldots,y_n)$? What happens to these densities as $n \to \infty$ (you may have to rescale and center).

1.8 Exercise
1. Above, we worked out the posterior for $\eta$ in the $(\alpha,\omega)$ family of conjugate priors

$$
\exp(\alpha(\omega \cdot \eta - \Lambda(\eta)))
$$

Write the parameters of $\pi(\eta|y)$ in terms of new $(\alpha,\omega)$.

2. Repeat the above for having observed $Y_1,\ldots,Y_n|\eta \overset{IID}{\sim} \text{P}_\eta$ where $\pi(\eta)d\eta = \exp(\alpha(\omega \cdot \eta - \Lambda(\eta)))d\eta$.

3. Suppose that instead $Y_1,\ldots,Y_n \overset{IID}{\sim} \text{P}_{\eta_0}$ for some fixed $\eta_0$. What do you expect will happen (as $n \to \infty$) to the sequence of densities

$$
\pi(\eta|y_1,\ldots,y_n)?
$$
1.9 Exercise

Suppose \( g_0(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} \) and \( t(y) = y \). That is, our family is

\[
P_{\eta}(dy) = e^{\eta y - \Lambda(\eta)} e^{-y^2/2} dy.
\]

The \texttt{R} package \texttt{sda} contains a gene expression data set comparing healthy to patients with prostate cancer.

```r
# R code
library(sda)
data(singh2002)
labels = singh2002$y
print(summary(labels))
extpression_data = singh2002$x
print(dim(expression_data))
```

1. Compute \( \hat{\ell}_0(y), \hat{\ell}_0(y) \).
2. Show that, for whatever prior one uses for \( \eta \):

\[
E(\eta|y) = y + \hat{\ell}(y)
\]
\[
\text{Var}(\eta|y) = 1 + \hat{\ell}(y).
\]
3. Form the 6033 two-sample \( t \)-statistics comparing healthy to normal, one for each gene. Using these \( t \)-statistics, form a density estimate and a numerical estimate of \( \hat{\ell} \) and \( \hat{\ell} \). (We will see shortly a more natural way to estimate the above derivatives...) Plot the estimated mean and variance as a function of \( y \).
4. Of course, \( t \)-statistics are not normally distributed. Transform your \( t \)-statistics to \( Z \) statistics using the \( t \) and \( N(0,1) \) distribution functions and repeat the above.

1.10 Exercise

Suppose we are in the Poisson family

\[
P_{\eta}(dy) = \exp(\eta \cdot y - \Lambda(\eta)) m_0(dy).
\]
1. For a given prior $\Pi(d\eta)$, what is the CGF for the exponential family $P(d\eta|y)$?

2. Compute $\ell_0(y), \bar{\ell}_0(y)$. Does it matter that $f, f_0$ are not densities with respect to Lebesgue measure?

1.11 Exercise

Suppose we observe a $2 \times 2$ contingency table when conducting a study on the efficacy of a new experimental surgery for ulcers.

<table>
<thead>
<tr>
<th></th>
<th>Success</th>
<th>Failure</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment</td>
<td>9</td>
<td>12</td>
<td>21</td>
</tr>
<tr>
<td>Control</td>
<td>7</td>
<td>17</td>
<td>24</td>
</tr>
<tr>
<td>Total</td>
<td>16</td>
<td>29</td>
<td>45</td>
</tr>
</tbody>
</table>

We would model this as $(Y_{11}, Y_{12}, Y_{21}, Y_{22}) \sim \text{Multinomial}(n, (\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}))$.

1. Show that given $(Y_1, Y_1, n)$ the only remaining “degrees of freedom” in determining the table is in any one of the entries, say $Y_{11}$.

2. The conditional mass function $Y_{11}|Y_1 = m, Y_1 = t$ is

$$
    g_\theta(Y_{11}) = \left( \frac{Y_{11}}{Y_{11}} \right)^{n - Y_1} \left( \frac{Y_1 - Y_{11}}{Y_1 - Y_{11}} \right)^{Y_{11} - \Lambda(\theta)} e^{\theta Y_{11} - \Lambda(\theta)}
$$

with range

$$
    \max(0, Y_1 + Y_1 - n) \leq Y_{11} \leq \min(Y_1, Y_1)
$$

where

$$
    \theta = \log \left( \frac{\pi_{11}}{\pi_{12}} \right) / \log \left( \frac{\pi_{21}}{\pi_{22}} \right).
$$

How would you estimate $\theta$ from the data?

3. Numerically compute and plot the likelihood and compare its maximum value to your estimate.

1.12 Exercise

Assume we are in the family

$$
    P_\eta(dx) = \exp \left( (\eta - 1) \log(x) - \log(\Gamma(\eta)) - x \right) \cdot 1_{[0, \infty)}(x) \, dx
$$

and you observe $x = 20$.

1. Find the maximum likelihood estimator, $\hat{\eta}(20)$. [Hint: you might need to look into digamma and/or trigamma functions. Here are two useful scipy links digamma, polygamma]

2. Estimate $\eta$ using the method of moments.