Random Walks, liquidity molasses and critical response in financial markets

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Introduction

- **Best known stylized fact in financial markets**: price changes are weakly correlated → approximate diffusion (Bachelier 1900)

- **Efficient market theory**: prices are fully rational and correspond to the best anticipation of future dividends → price changes can only be due to unpredictable news; **but**: excess volatility, with long range, multiscale memory!

- **More fundamentally**: ambiguous information, psychological and cognitive biases, herding (cf. prediction of financial analysts!)

- **‘Zero intelligence’ investors**: Each trade has a totally random motivation, but has a non zero impact on the price – each trade is considered by others as containing *some* information
Volatility clustering: comparison between the Dow Jones and a Brownian Random Walk

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Volatility correlation

Data (Averaged over 500 stocks)

Fit, $\nu=0.22$

Fit, $\exp(-t/4)$

Fit, $\exp(-t/40)$

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Power-law response to volatility shocks - HF

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Analysts herding behaviour

With O. Guedj

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Empirical facts on trades and quotes data


- Example of a liquid stock: France Telecom – 5000 trades/day – 1.2 M-trades in 2002

- Quotes: Bid price + Ask price → midpoint \( m = (\text{Bid} + \text{Ask})/2 \)

- Trades: At the Ask → \( \varepsilon = +1 \), at the Bid → \( \varepsilon = -1 \)
The order book

### LAST MATCH
- **Price**: 25.1290
- **Time**: 11:42:15.597

### TODAY'S ACTIVITY
- **Orders**: 67,212
- **Volume**: 12,778,400

#### BUY ORDERS

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As of 11:42:15.769

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Order flow and order book: New ‘Stylized facts’

- Many new quantities can be analyzed
  - Statistics of the ‘rain’ of incoming orders as a function of distance from current bid/ask
  - Average size of the queue as a function of distance from current bid/ask
  - Probability distribution of the size of the queue
  - Collective modes of the order book

- Also: interaction between order book and price changes, between order flow and price changes (‘Impact’) – see below.
Statistics of the rain of orders

- As a function of the distance $\Delta$ from the current bid/ask:
  - Probability that a new order is placed is very broad – up to 50% away from current price!
  - Power law distribution $P(\Delta) \approx \Delta^{-1-\mu}$ with:
    * $\mu \sim 0.6$ for (liquid) CAC40 stocks
    * $\mu \sim 1.$ for (liquid) NASDAQ stocks
    * $\mu \sim 1.5$ for LSE stocks (Farmer & Zovko)
  - Conditional average size of the order: $\langle \Phi \rangle \approx \Phi_0$ for $\Delta \leq \Delta^*$, $\langle \Phi \rangle \approx \Delta^{-\nu}$ for $\Delta \geq \Delta^*$, with $\nu \sim 1.5$

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Statistics of the rain of orders

Note: same distribution for buy and sell orders

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Shape of the order book

- As a function of the distance $\Delta$ from the current bid/ask:
  - The average size of the queue $\rho(\Delta)$ has a characteristic ‘humped’ shape, with a maximum away from the bid (ask)
  - Symmetric shape for buy and sell orders
  - The shape is found to be stock independent for French stocks
  - The shape can be different on NASDAQ stocks – but not a centralized market!
The shape of the order book

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A simple analytical model I

- Orders at distance $\Delta$ at time $t$ are those which were placed there at a time $t' < t$, and have survived until time $t$, that is:
  - (i) have not been cancelled;
  - (ii) have not been touched by the price at any intermediate time $t''$ between $t'$ and $t$.

- Therefore:

$$\rho(\Delta, t) = \int_{-\infty}^{t} dt' \int du P(\Delta + u) \mathcal{P}(u|C(t, t')) e^{-\Gamma(t-t')} ,$$

where $\mathcal{P}(u|C(t, t'))$ is the conditional probability for the ask difference $u = a(t) - a(t')$, such that $\Delta + a(t) - a(t'') \geq 0$, $\forall t'' \in [t', t]$.

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A simple analytical model II

- Assuming that the price follows a Gaussian random walk:

\[
\rho_{st}(\Delta) = e^{-\alpha \Delta} \int_{0}^{\Delta} du P(u) \sinh(\alpha u) + \sinh(\alpha \Delta) \int_{\Delta}^{\infty} du P(u) e^{-\alpha u},
\]

where \( \alpha^{-1} = \sqrt{D/2\Gamma} \) measures the typical variation of price during the lifetime of an order.

- When \( \mu < 1 \), \( \alpha \) can be rescaled away, and:

\[
\rho_{st}(\tilde{\Delta}) = e^{-\tilde{\Delta}} \int_{0}^{\tilde{\Delta}} du u^{-1-\mu} \sinh(u) + \sinh(\tilde{\Delta}) \int_{\tilde{\Delta}}^{\infty} du u^{-1-\mu} e^{-u}
\]

with \( \tilde{\Delta} = \alpha \Delta \).

Note: for \( \Delta \to 0 \), \( \rho_{st}(\Delta) \propto \Delta^{1-\mu} \to 0 \to \text{hump} \)!

- Reproduces the numerical results satisfactorily

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Comparison numerical model - analytical approx.
Price dynamics: Diffusion

• **Price fluctuations** in trade time:

\[ \mathcal{D}(\ell) = \left\langle (p_{n+\ell} - p_n)^2 \right\rangle \approx D\ell \]

• Note: \( \sqrt{\mathcal{D}(1)} \approx 0.01 \) Euros: precisely the bid-ask spread.
  True for all stocks.

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Price Diffusion

![Graph showing price diffusion over time with different markers for FT (2001 - 1st semester), FT (2001 - 2nd semester), and FT (2002).]
Price Diffusion

![Graph showing price diffusion over time for different companies: Pechiney, Vivendi, Barclays, TF1. The y-axis represents \( D(\Delta l)^{1/2} \) in arbitrary units, and the x-axis represents time in trades.](image)
Price dynamics: Response function/Market impact

- **Average response function:**
  \[
  \mathcal{R}(\ell) = \left\langle \left( p_{n+\ell} - p_n \right) \cdot \varepsilon_n \right\rangle
  \]
  Weak growth as a function of \( \ell \) and then declines for \( \ell > \ell^* \)

- **Response to a trade of volume \( V \):**
  \[
  \mathcal{R}(\ell, V) = \left\langle \left( p_{n+\ell} - p_n \right) \cdot \varepsilon_n \right\rangle \bigg|_{V_n=V}.
  \]
  Approximate factorisation: \( \mathcal{R}(\ell, V) \approx \ln V \times \mathcal{R}(\ell) - \) large volumes affect prices less than small volumes! (cf. Hasbrouck (1991), Gopikrishnan et al., Lillo et al.)

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Average response

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Average response

![Graph of R(l) vs. l]

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Response: factorisation
Price dynamics: Fraction of informed trades

- Full distribution of $u_\ell = (p_{n+\ell} - p_n) \varepsilon_n$:

  $$\mathcal{R}(\ell) = \langle u_\ell \rangle \quad \mathcal{D}(\ell) = \langle u_\ell^2 \rangle$$

- Only very small asymmetry that disappears when $u_\ell$ is shifted by 0.01 Euros; skewness decays as $\ell^{-1/2}$.

- Very few trades can be qualified as ‘informed’, i.e. correctly anticipating short term moves to at least cover minimal costs (cf. *Do investors trade too much?* – Odean 1999)

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Impact distribution

$l=128$

- solid line: unshifted
- dashed line: $u_i$ shifted by 0.01 Euros

$u_i < 0$

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Skewness

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Price dynamics: A fluctuation-response relation

- For Brownian random walks: Mobility = Diffusion/Temperature

- Similar relation in financial markets? Rosenow 2001

\[
\frac{D(\ell)}{\ell} = A R^2(\ell) + B
\]
Fluctuation-Response Relation

FT 2002
FT 2001

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Market order flow: Long term memory

- **Trade correlations:**
  \[ C(\ell) = \langle \epsilon_{n+\ell} \epsilon_n \rangle \approx \frac{C_0}{\ell^\gamma} \]
  with \( \gamma < 1 \) (\( \gamma \approx 1/4 \) for FT, \( \approx 1/2 \) for Vodafone – see Lillo-Farmer)

- **Paradox:** The effective number of identical trades grows with \( \ell \):
  \[ N_e \approx 1 + \sum_{\ell=1}^{1000} C_0(\ell) \approx 1 + \frac{C_0}{1 - \gamma} 1000^{1-\gamma} \approx 50 \]

- **\( R(\ell) \)** should increase by a large factor and one should observe superdiffusion.

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Trade correlations

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A micro-model of price fluctuations

- Linear superposition of impacts:

\[ p_n = \sum_{n' < n} G_0(n - n') \varepsilon_{n'} \ln V_{n'} + \sum_{n' < n} \eta_{n'}, \]

where \( G_0(.) \) is the ‘bare’, non permanent response function (or propagator) of a single trade.

- Alternative model – Lillo-Farmer:

\[ p_n = \sum_{n' < n} \frac{\varepsilon_{n'} V_{n'}^{\beta}}{\lambda_{n'}} + \sum_{n' < n} \eta_{n'} : \]

permanent, but fluctuating impact depending on instantaneous liquidity – see discussion and comparison in cond-mat/0406224

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A simple case first

- Simple case: no correlation in signs

\[ R(\ell) \sim G_0(\ell) \]

\[ D(\ell) \sim \left( \sum_{0 < n \leq \ell} G_0^2(n) + \sum_{n > 0} [G_0(\ell + n) - G_0(n)]^2 \right), \]

- For a permanent impact: Constant response and pure diffusion
Role of correlations

- More generally:

\[ R(\ell) = \langle \ln V \rangle G_0(\ell) + \sum_{0<n<\ell} G_0(\ell-n)C_1(n) + \sum_{n>0} [G_0(\ell + n) - G_0(n)] C_1(n) \]

(and a more complicated equation for \( D(\ell) \)).

- If \( G_0 \) were constant, then \( R(\ell) \propto \ell^{1-\gamma} \) and \( D(\ell) \propto \ell^{2-\gamma} \)

- Only way out: the impact of single trades is itself non-permanent

\[ G_0(n) = \frac{R_0}{(n_0 + n)^\beta} \]

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Role of correlations

- Asymptotic behaviour:
  \[ D(\ell) \sim \ell^{2-2\beta-\gamma}, \quad R(\ell) \sim \ell^{1-\beta-\gamma} \]

- For diffusion to be normal: \( \beta = (1 - \gamma)/2 \approx 3/8 \)

- but \( R(\ell) \sim \ell^{1-3/8-1/4} \sim \ell^{3/8} \) incompatible with data ??

- In fact:
  \[ R(\ell) \sim \frac{\Gamma(1 - \gamma)}{\Gamma(\beta)\Gamma(2 - \beta - \gamma)} \left[ \frac{\pi}{\sin \pi \beta} - \frac{\pi}{\sin \pi (1 - \beta - \gamma)} \right] \ell^{1-\beta-\gamma} \]
Theoretical and empirical response function

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Theoretical and empirical response function

\[ G(l) \]

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Interpretation: two antagonist categories of traders

- **Liquidity takers**: place market orders, as a result of true/putative information, or urge to buy/sell. Must limit their impact → orders are cut in small pieces and create serial correlations due to their size.

- **Liquidity providers**: place limit orders, but no long term positions in markets. Must limit the fluctuations of the price → slow mean reversal force: *liquidity molasses*. How: order ‘barrier’ at the ask + anticorrelated quotes.

- Both populations compete such as to remove arbitrage opportunities (linear correlations), and impose $\beta \approx (1 - \gamma)/2$.

- Volatility may come from these trading rules alone, and only weakly from external news.

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Proximity of the critical line

Fit parameters

\[ \beta = \frac{1 - \gamma}{2} \]
Conclusion: a critical dynamical equilibrium

- Price diffusion: result from a subtle competition (compensation) between persistent effects (liquidity takers, correlated orders) and antipersistent effects (liquidity providers, mean reverting forces).

- Both effects are characterized by scale-less, power-law functions of time.

- Dynamical equilibrium between the two can be temporarily broken → large, intermittent fluctuations and crashes.